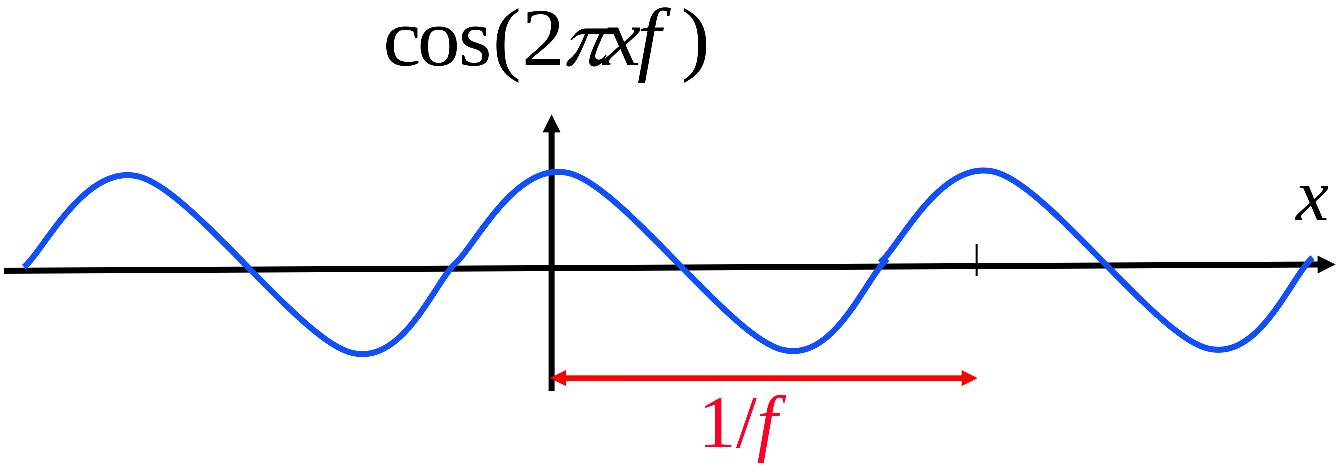
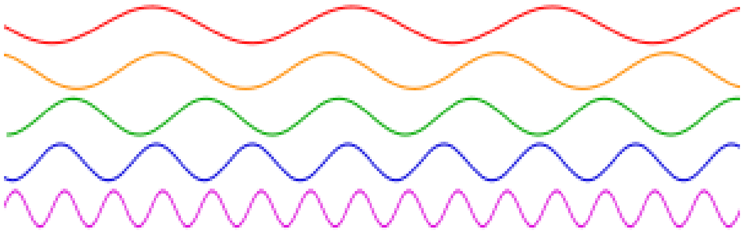
4. Fourier-transzformáció

Frekvencia és hullámhossz



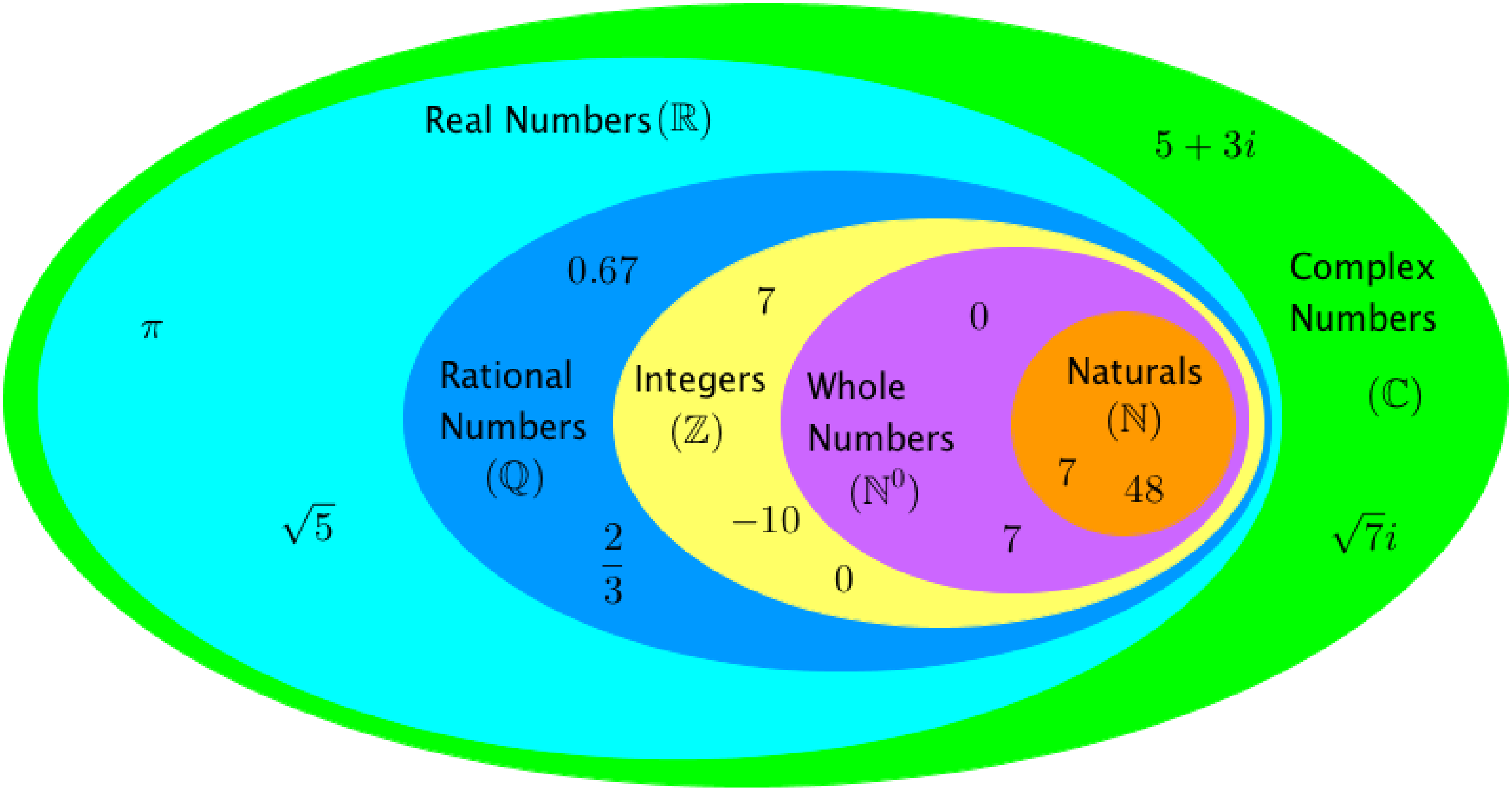


amplitúdó: 1

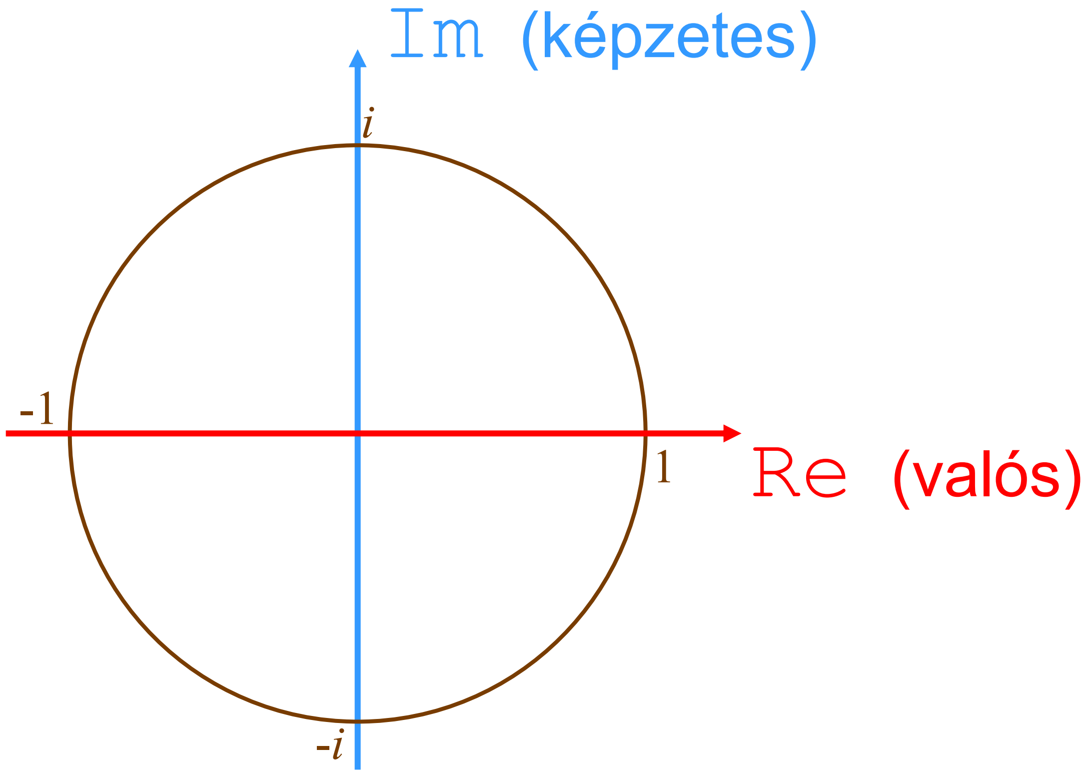
hullámhossz: $1/f$

frekvencia: f

Számok

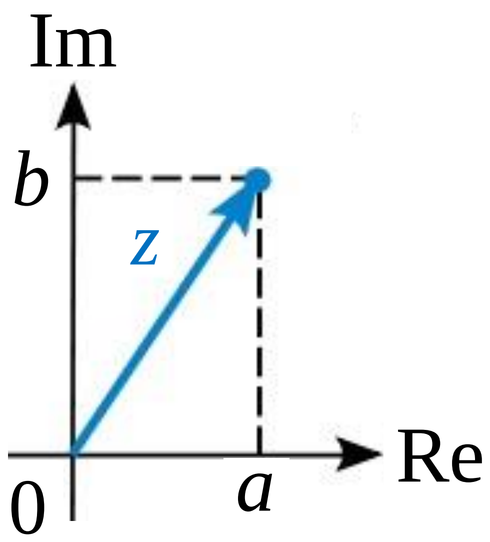

Real Numbers $(\mathbb{R})$
$5+3i$
0.67
7
0
Complex Numbers $(\mathbb{C})$
π
Rational Numbers $(\mathbb{Q})$
Integers $(\mathbb{Z})$
Whole Numbers $(\mathbb{N}^0)$
Naturals $(\mathbb{N})$
7
48
$\sqrt{5}$
$\frac{2}{3}$
-10
7
0
$\sqrt{7}i$

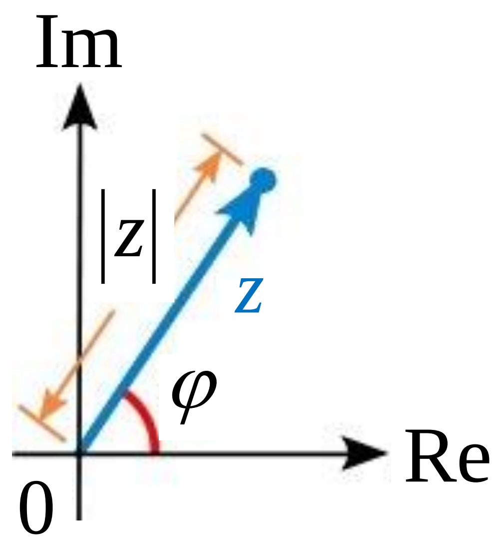
Komplex számok



$i^2 = -1$ imaginárius egység

Komplex számok





$$z = a + b \cdot i$$

$$z = |z| \cdot \left(\cos(\varphi) + i \cdot \sin(\varphi)\right)$$

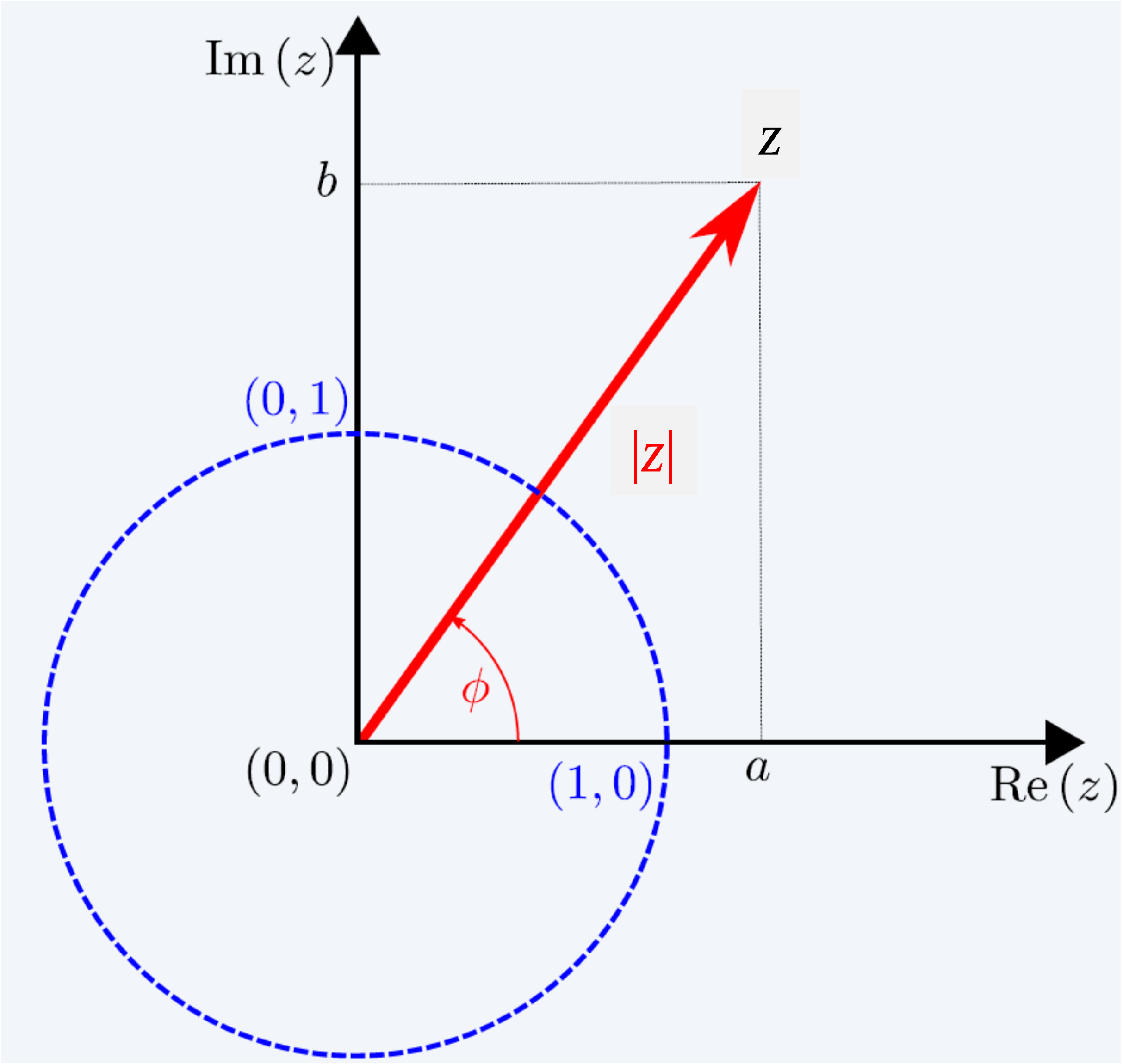
$$\mathrm{Re}(z) = a = |z| \cdot \cos(\varphi)$$

$$\mathrm{Im}(z) = b = |z| \cdot \sin(\varphi)$$

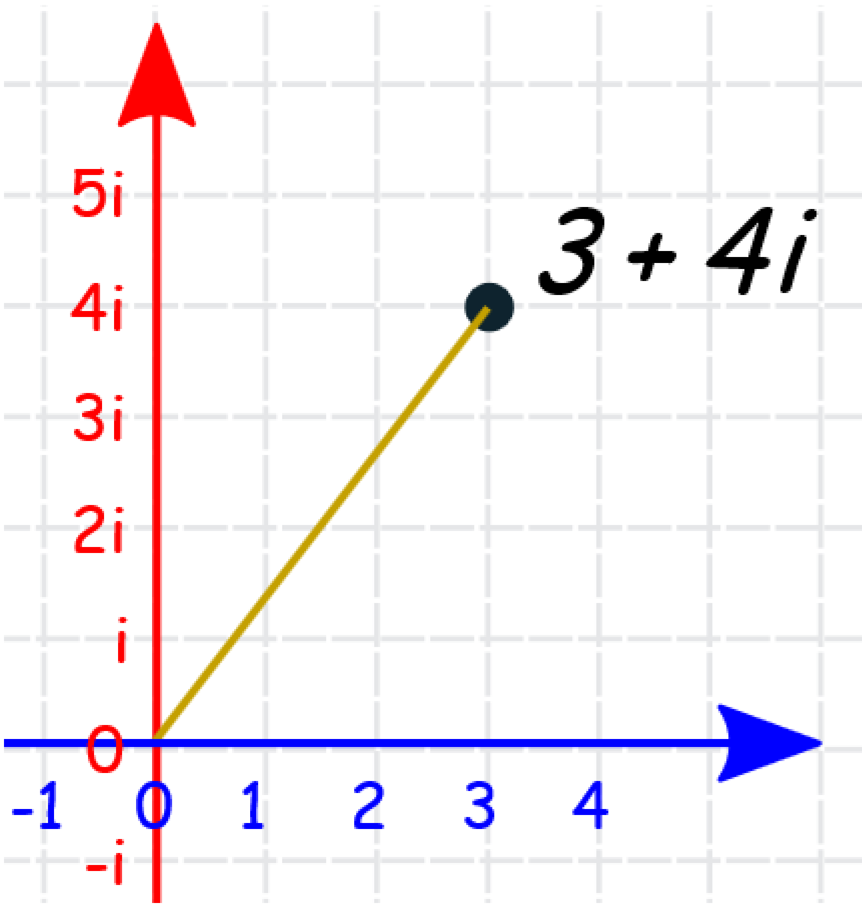
$$|z| = \sqrt{a^2 + b^2}$$

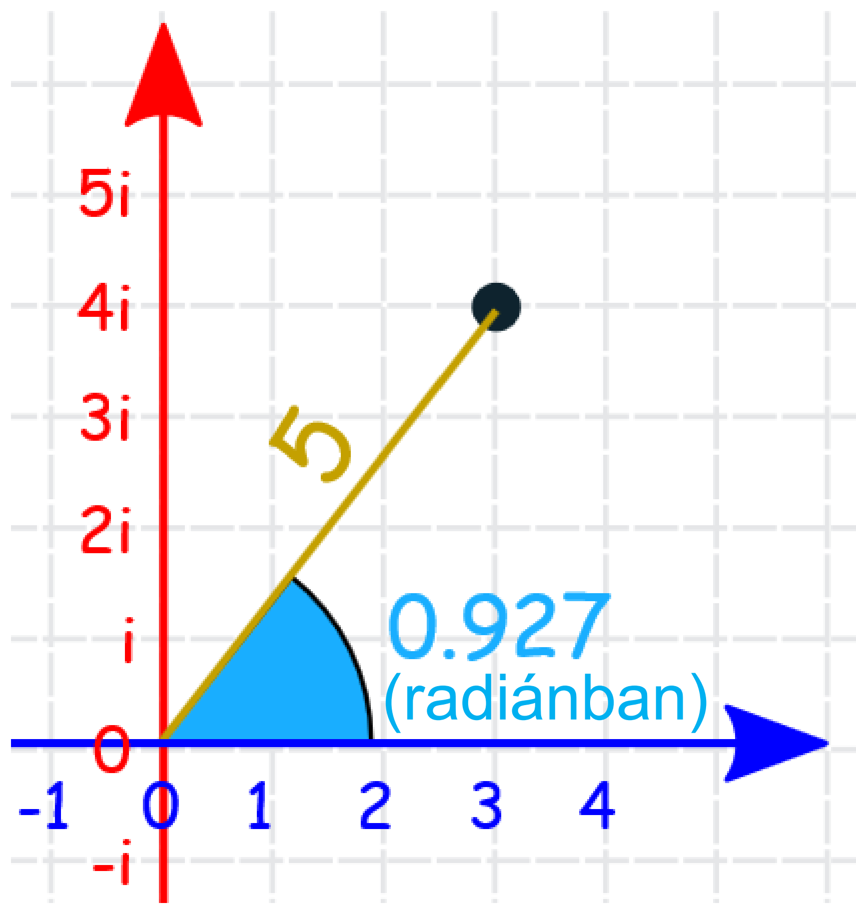
$$\varphi = \tan^{-1}(b/a)$$

Komplex számok



Komplex számok





Komplex számok

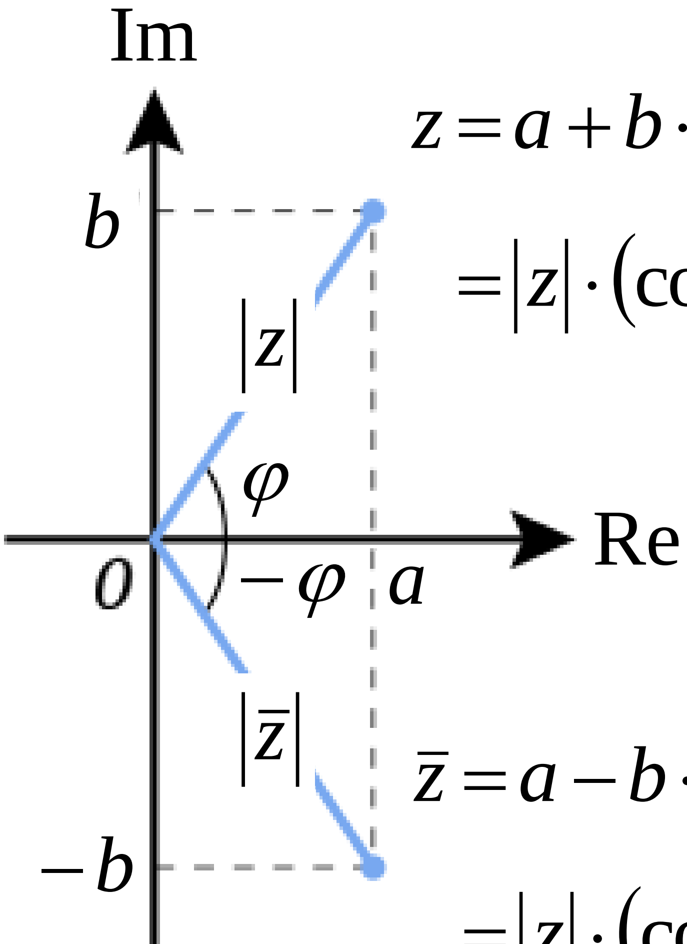
$$\mathrm{Re}(z) = a = |z| \cdot \cos(\varphi)$$ valós rész

$$\mathrm{Im}(z) = b = |z| \cdot \sin(\varphi)$$ képzetes rész

$$|z| = \sqrt{a^2 + b^2}$$ abszolút érték / magnitúdó

$$\varphi = \tan^{-1}(b / a)$$ fázis szög

Komplex konjugált



$$z = a + b \cdot i$$

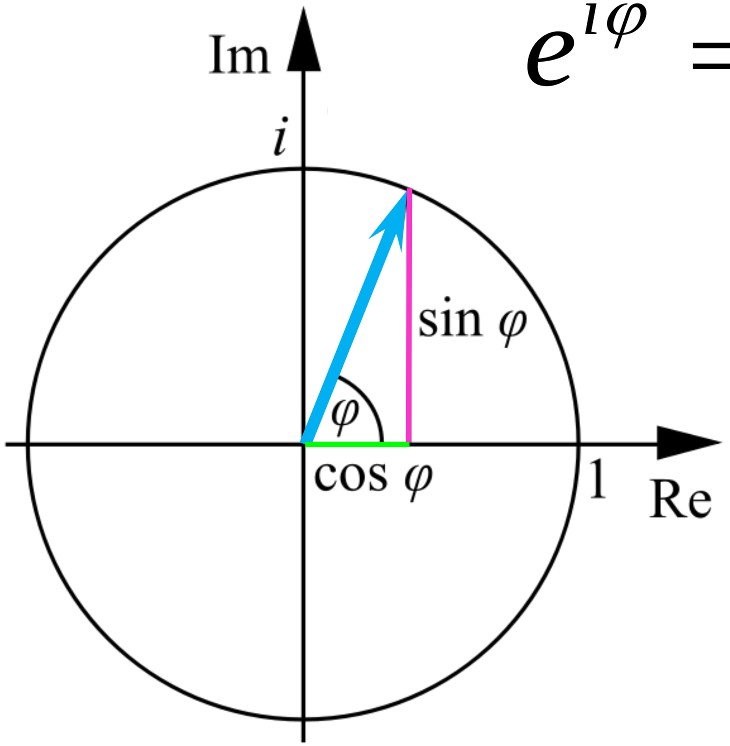
$$= |z| \cdot (\cos(\varphi) + i \cdot \sin(\varphi))$$

$$\bar{z} = a - b \cdot i$$

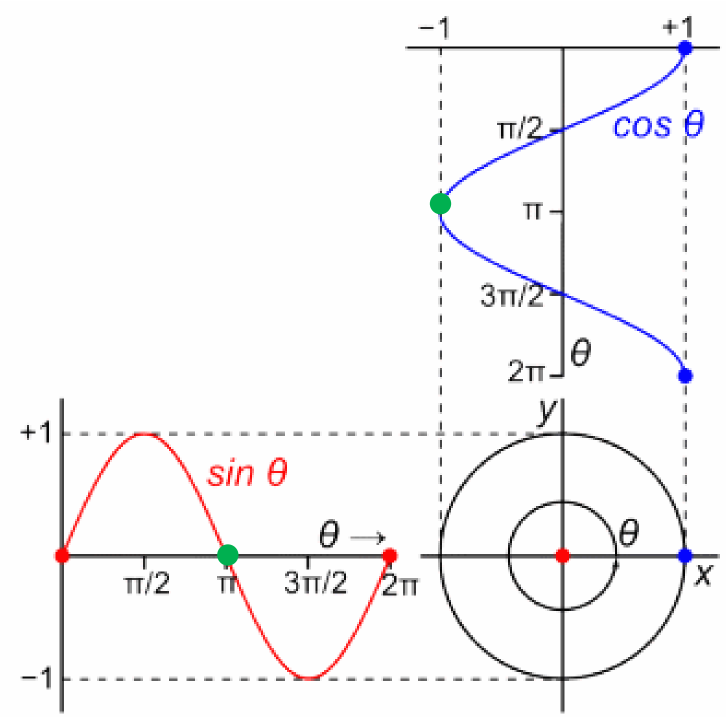
$$= |z| \cdot (\cos(\varphi) - i \cdot \sin(\varphi))$$

Euler-formula

$$e^{i\varphi} = \cos(\varphi) + i \cdot \sin(\varphi)$$



Euler-formula



$$e^{i\pi} = -1$$

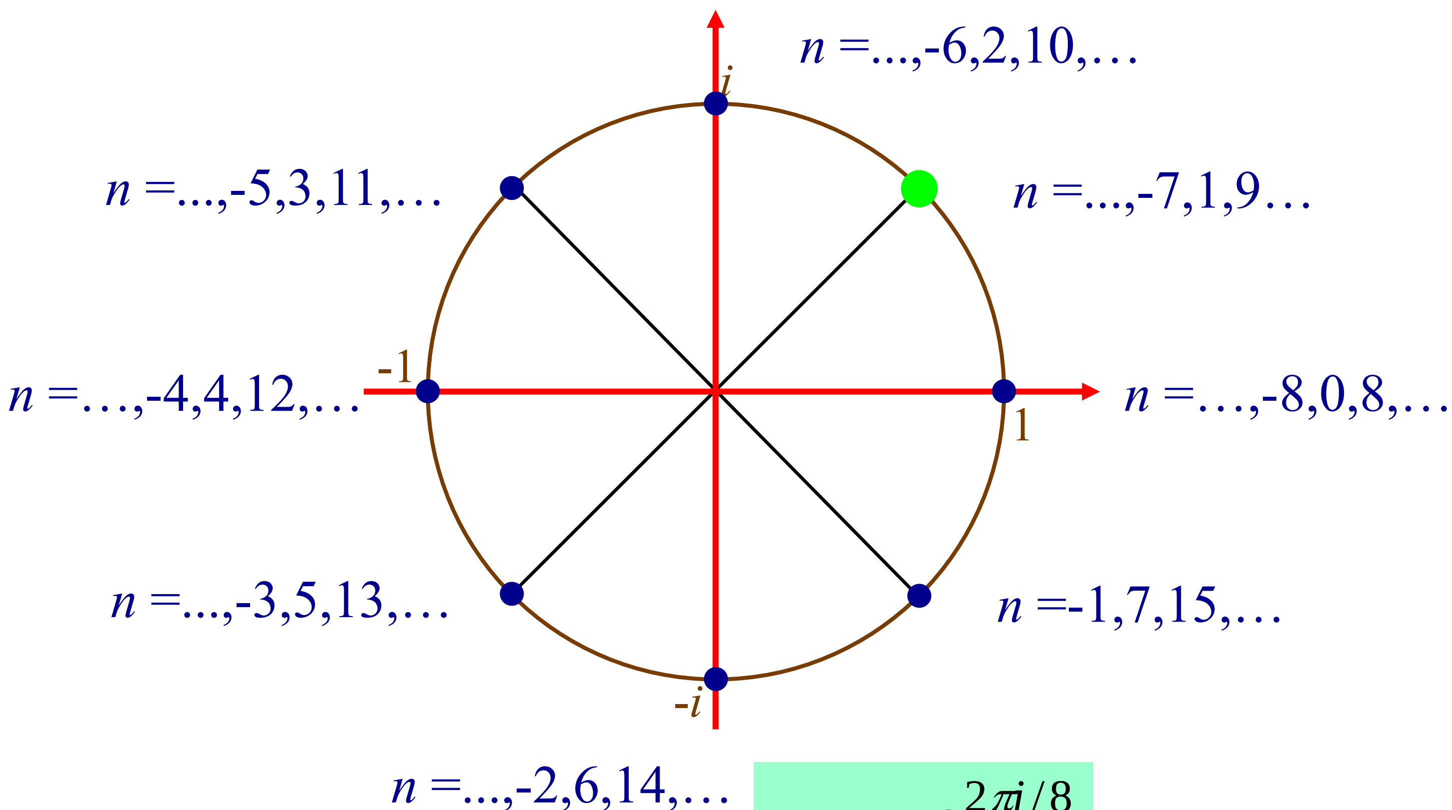
$$e^{i\pi} + 1 = 0$$

$$e^{i\pi} = \cos(\pi) + i \cdot \sin(\pi) = -1 + 0 = -1$$

Komplex számok hatványozása

$$\begin{aligned}\left(e^{i\varphi}\right)^n &= \left(\cos(\varphi)+i\cdot\sin(\varphi)\right)^n \\ &= \cos(n\varphi)+i\cdot\sin(n\varphi) \\ &= e^{in\varphi}\end{aligned}$$

Egységgyökök és hatványaik



$$\omega_8 = e^{2\pi i/8}$$

$$\omega_8^n = e^{2\pi i n/8} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

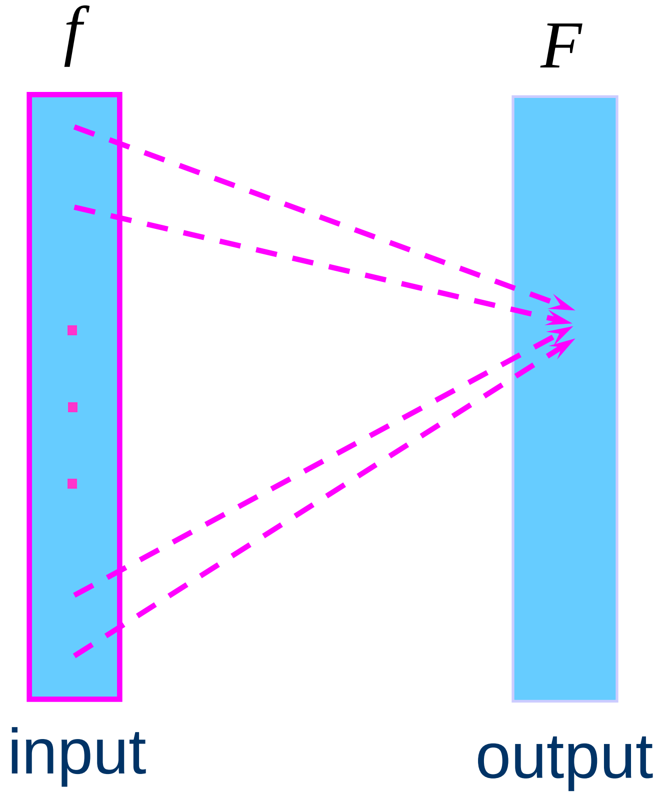
Alkatrészek és termékek

	Mojito	Tequila Sunrise	Daiquiri	Margarita
fehér rum	5 cl	-	5 cl	-
tequila	-	4 cl	-	4 cl
triple sec	-	-	-	2 cl
lime leve	3 cl	-	2 cl	2 cl
grenadine	-	2 cl	-	-
menta	2 ág	-	-	-
cukorszirup	1 ek	-	1 ek	-
narancslé	-	1,5 dl	-	-
szóda	1,5 dl	-	-	-
jégkása	-	-	1,5 dl	1,5 dl

Alkatrészek és termékek

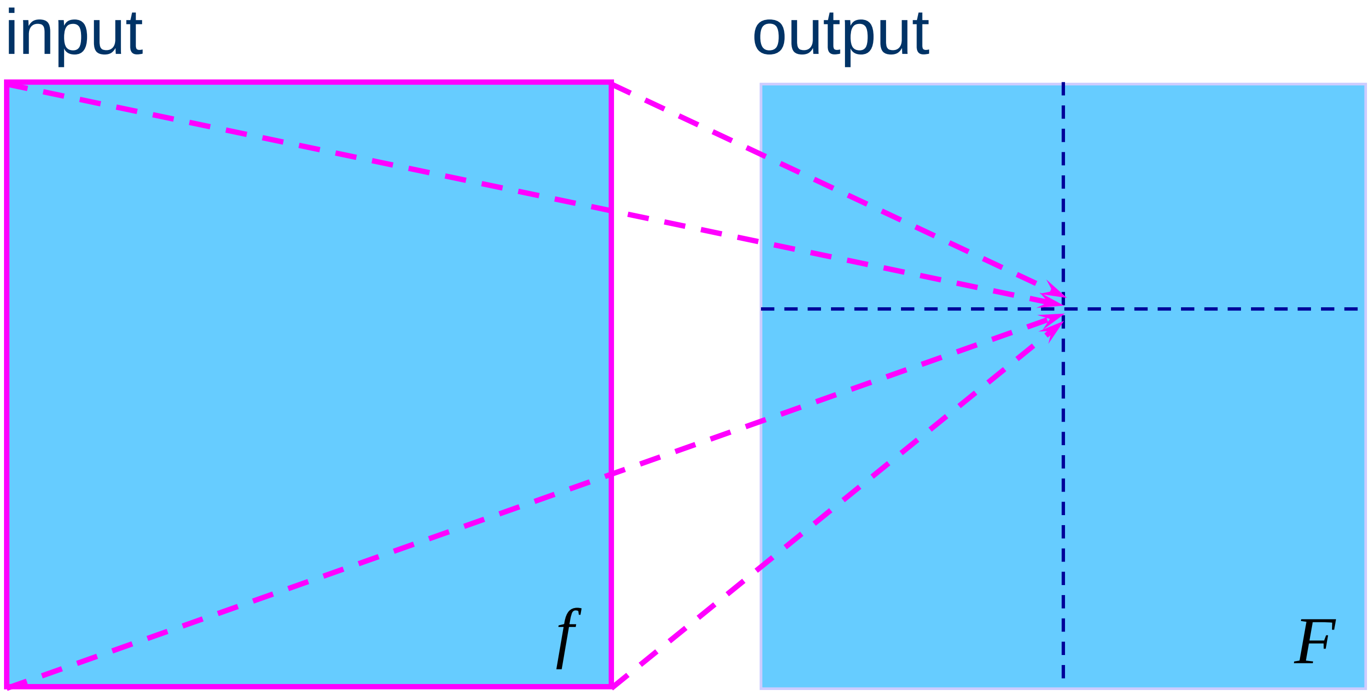
bázisfüggvények	súlyok			
fehér rum	5 cl	-	5 cl	-
tequila	-	4 cl	-	4 cl
triple sec	-	-	-	2 cl
lime leve	3 cl	-	2 cl	2 cl
grenadine	-	2 cl	-	-
menta	2 ág	-	-	-
cukorszirup	1 ek	-	1 ek	-
narancslé	-	1,5 dl	-	-
szóda	1,5 dl	-	-	-
jégkása	-	-	1,5 dl	1,5 dl

Globális művelet – 1D



az eredmény minden egyes eleme függ az input összes komponensétől

Globális művelet – 2D



az eredmény minden egyes eleme
függ az input összes komponensétől

Transzformáció – 1D

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-1) \end{bmatrix} \Leftrightarrow \begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(M-1) \end{bmatrix}$$

Lineáris transzformáció – 1D

$$\longrightarrow \qquad F(X) = \sum_{x=0}^{M-1} f(x) \cdot w(x, X)$$

súlyok

bázis-
függvények

$$\longleftarrow \qquad f(x) = \sum_{X=0}^{M-1} F(X) \cdot w'(x, X)$$

$$(x, X = 0, 1, \ldots, M-1)$$

Transzformáció – 2D

$$
\begin{bmatrix}
f(0,0) & f(0,1) & \cdots & f(0,N-1) \\
f(1,0) & f(1,1) & \cdots & f(1,N-1) \\
\vdots & \vdots & \ddots & \vdots \\
f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1)
\end{bmatrix}
$$

$$\Updownarrow$$

$$
\begin{bmatrix}
F(0,0) & F(0,1) & \cdots & F(0,N-1) \\
F(1,0) & F(1,1) & \cdots & F(1,N-1) \\
\vdots & \vdots & \ddots & \vdots \\
F(M-1,0) & F(M-1,1) & \cdots & F(M-1,N-1)
\end{bmatrix}
$$

Lineáris transzformáció – 2D

$$\rightarrow \quad F(X,Y) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \cdot w(x,y,X,Y)$$

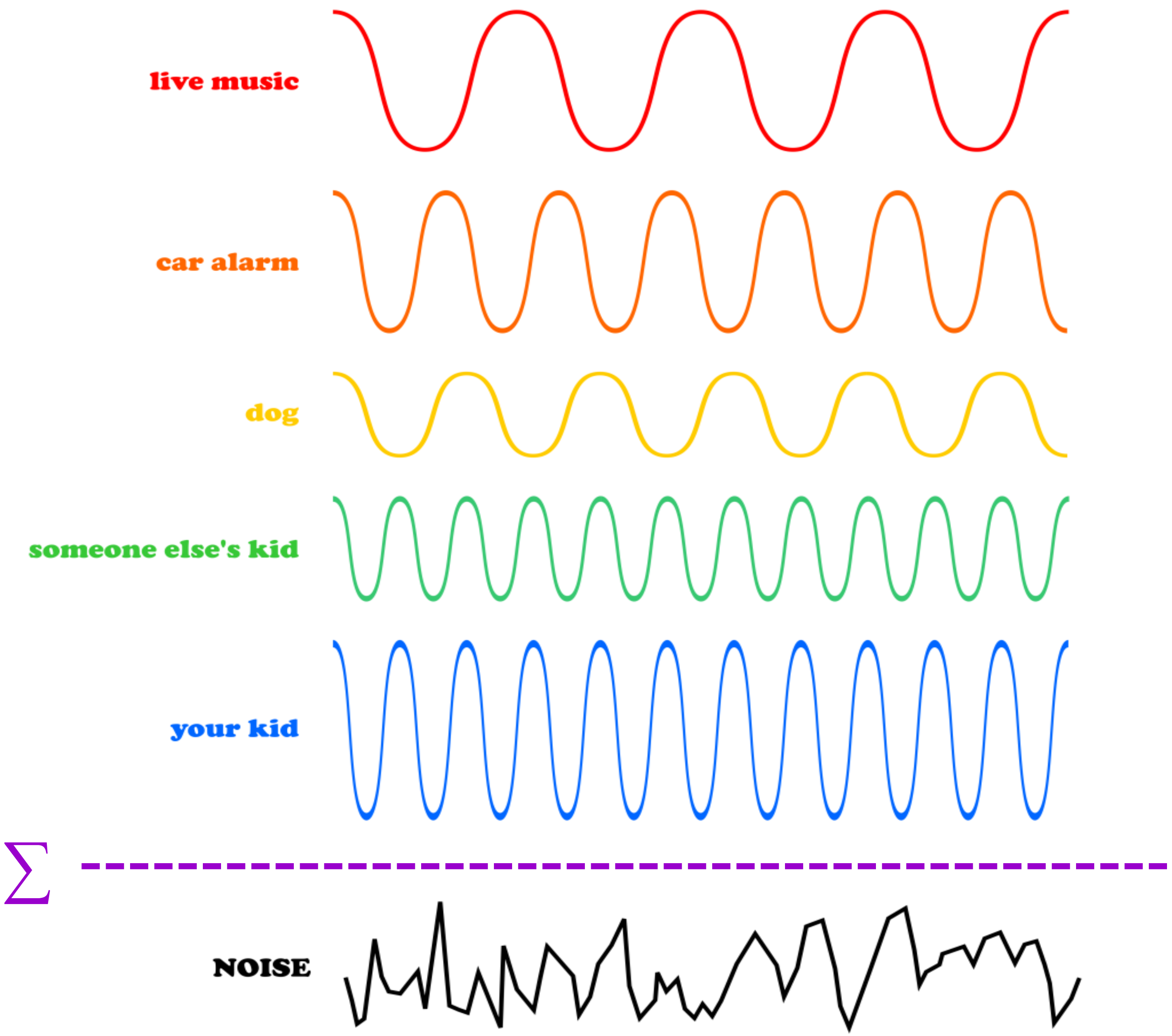
súlyok

bázis-
függvények

$$\leftarrow \quad f(x,y) = \sum_{X=0}^{M-1}\sum_{Y=0}^{N-1} F(X,Y) \cdot w'(x,y,X,Y)$$

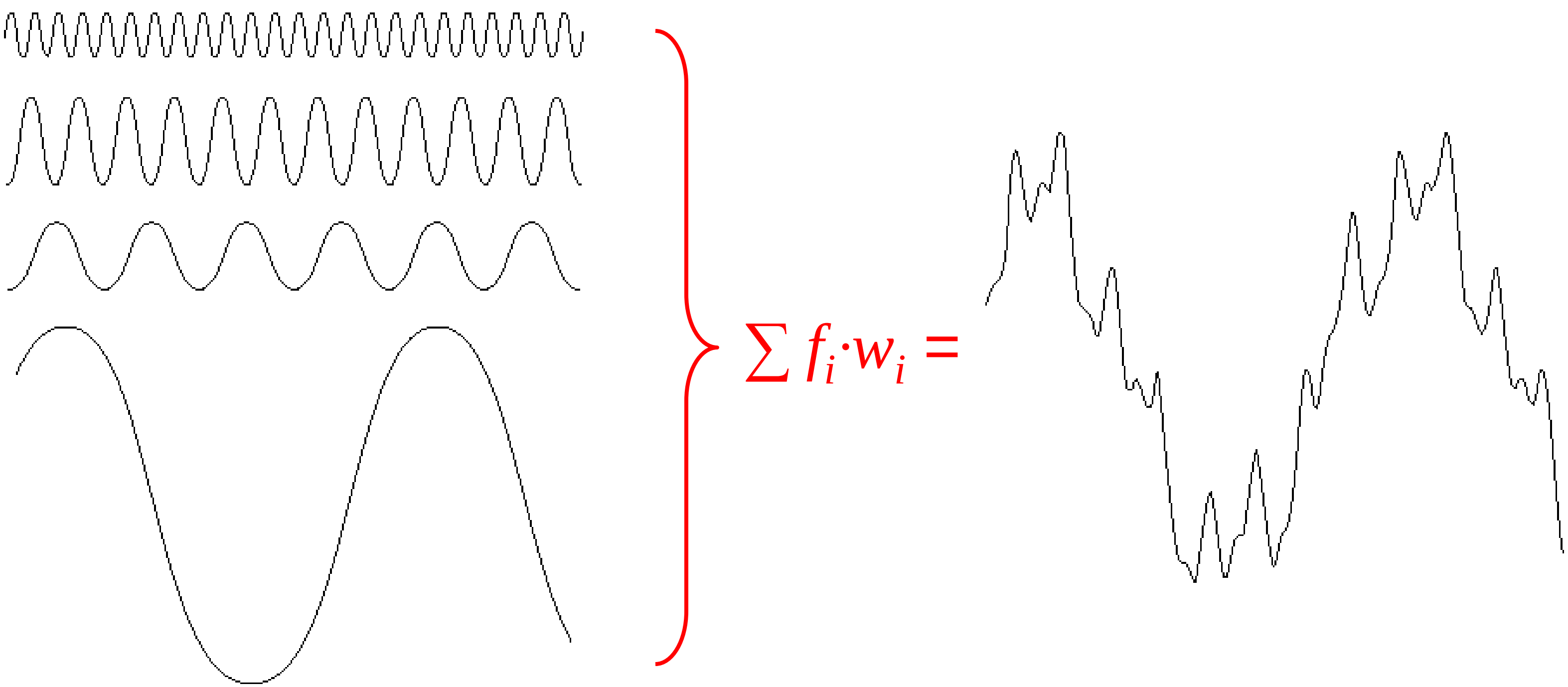
$$\left(\begin{array}{l} x, X = 0, 1, \ldots, M-1; \\ y, Y = 0, 1, \ldots, N-1 \end{array} \right)$$

Összegzett jelek



Frekvencia-analízis

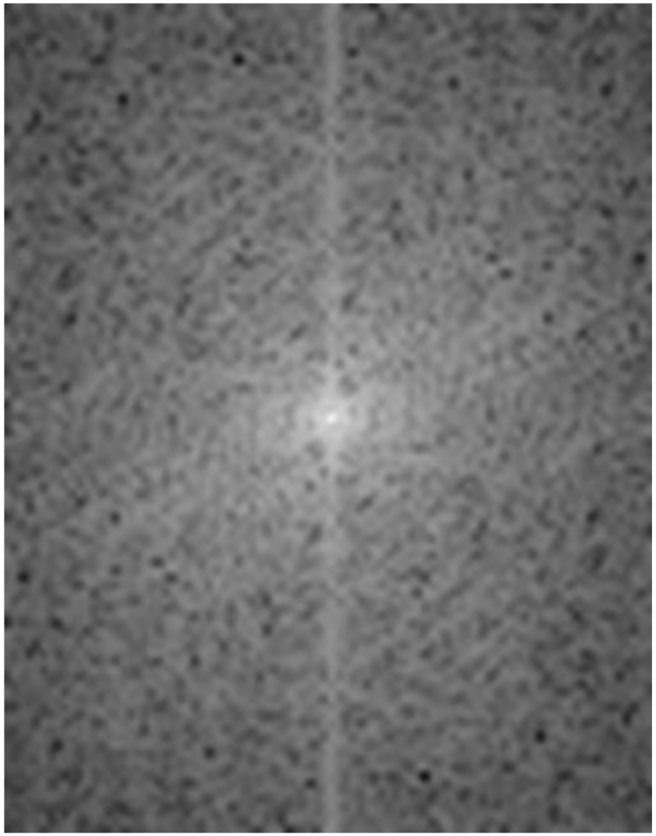
Nem-periodikus függvények is felírhatók különböző frekvenciájú $\sin$ és $\cos$ függvények súlyozott összegeként.
Ennek matematikai eszköze a **Fourier-transzformáció**.



Fourier-transzformáció

Jean Baptiste Joseph Fourier

↔



1D Fourier-transzformáció

DFT

$$F(X) = \sum_{x=0}^{M-1} f(x) \cdot w(x, X) \qquad (X = 0,1,\ldots, M-1)$$

$$w(x, X) = \frac{1}{M} e^{-2\pi i \frac{xX}{M}}$$

IDFT

$$f(x) = \sum_{X=0}^{M-1} F(X) \cdot w'(x, X) \qquad (x = 0,1,\ldots, M-1)$$

$$w'(x, X) = e^{2\pi i \frac{xX}{M}}$$

$$(x, X = 0, 1, \ldots, M-1)$$

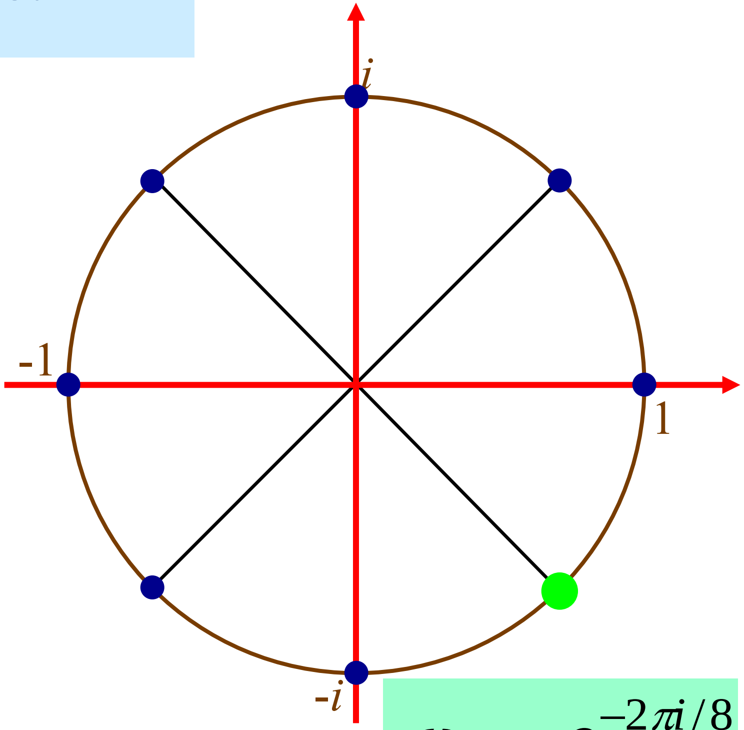
Diszkrét Fourier-transzformáció (DFT) – 1D

$\omega_M = e^{-2\pi i/M}$ egy M-edik egységgyök

$$w(x,X) = \frac{1}{M} e^{-2\pi i \frac{xX}{M}} = \frac{1}{M} \omega_M^{xX}$$

$$w'(x,X) = e^{2\pi i \frac{xX}{M}} = \overline{\omega}_M^{xX}$$

$$(x, X = 0,1,\ldots,M-1)$$



$\omega_8 = e^{-2\pi i/8}$

Diszkrét Fourier-transzformáció (DFT) – 1D

$$F(X) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\cdot e^{-2\pi i\frac{xX}{M}} = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\cdot \omega_M^{xX} \qquad (X = 0,1,\ldots,M-1)$$

$$\omega_M = e^{-2\pi i/M}$$

$$\underbrace{\frac{1}{M}\begin{bmatrix} \omega_M^0 & \omega_M^0 & \omega_M^0 & \cdots & \omega_M^0 \\ \omega_M^0 & \omega_M^1 & \omega_M^2 & \cdots & \omega_M^{M-1} \\ \omega_M^0 & \omega_M^2 & \omega_M^4 & \cdots & \omega_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_M^0 & \omega_M^{M-1} & \omega_M^{2(M-1)} & \cdots & \omega_M^{(M-1)(M-1)} \end{bmatrix}}_{A} \cdot \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix} = \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(M-1) \end{bmatrix}$$

Inverz Diszkrét Fourier-transzformáció (IDFT) – 1D

$$f(x) = \sum_{X=0}^{M-1} F(X) \cdot e^{2\pi i \frac{xX}{M}} = \sum_{X=0}^{M-1} F(X) \cdot \overline{\omega}_M^{xX} \qquad (x = 0, 1, \ldots, M-1)$$

$$\underbrace{\begin{bmatrix} \overline{\omega}_M^{0} & \overline{\omega}_M^{0} & \overline{\omega}_M^{0} & \cdots & \overline{\omega}_M^{0} \\ \overline{\omega}_M^{0} & \overline{\omega}_M^{1} & \overline{\omega}_M^{2} & \cdots & \overline{\omega}_M^{M-1} \\ \overline{\omega}_M^{0} & \overline{\omega}_M^{2} & \overline{\omega}_M^{4} & \cdots & \overline{\omega}_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\omega}_M^{0} & \overline{\omega}_M^{M-1} & \overline{\omega}_M^{2(M-1)} & \cdots & \overline{\omega}_M^{(M-1)(M-1)} \end{bmatrix}}_{B} \cdot \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(M-1) \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix}$$

$$\omega_M = e^{-2\pi i / M}$$

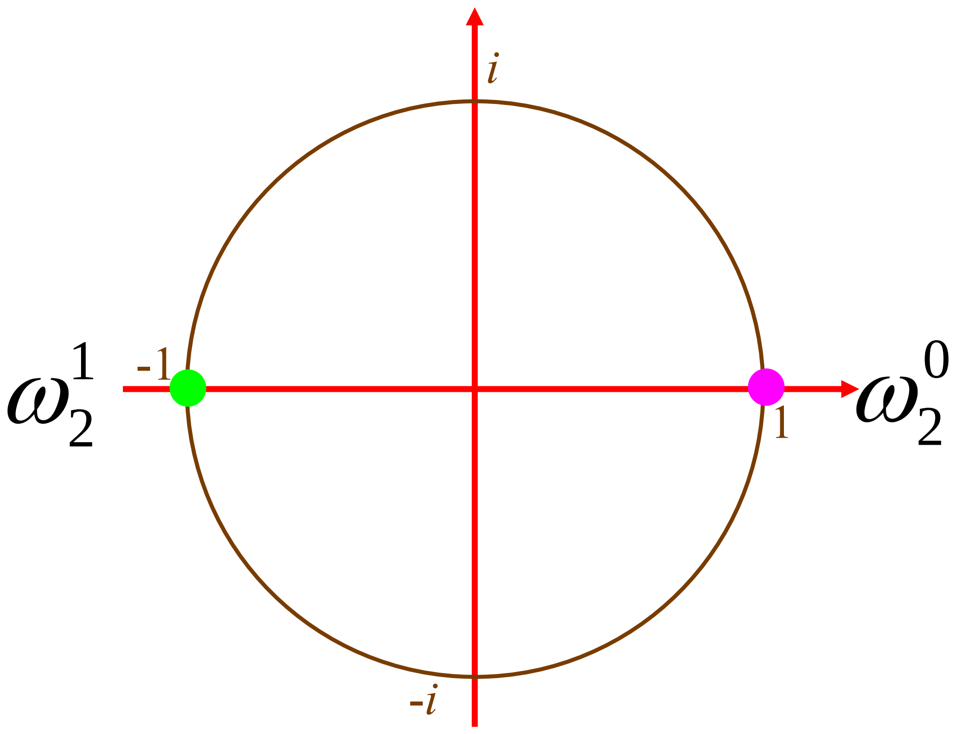
A DFT és az IDFT mátrixai

$$A = A^T = \frac{1}{M} \cdot \overline{B} = B^{-1} \qquad \frac{1}{M}\begin{bmatrix} \omega_M^0 & \omega_M^0 & \omega_M^0 & \cdots & \omega_M^0 \\ \omega_M^0 & \omega_M^1 & \omega_M^2 & \cdots & \omega_M^{M-1} \\ \omega_M^0 & \omega_M^2 & \omega_M^4 & \cdots & \omega_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_M^0 & \omega_M^{M-1} & \omega_M^{2(M-1)} & \cdots & \omega_M^{(M-1)(M-1)} \end{bmatrix}$$

$$B = B^T = M \cdot \overline{A} = A^{-1} \qquad \begin{bmatrix} \overline{\omega}_M^0 & \overline{\omega}_M^0 & \overline{\omega}_M^0 & \cdots & \overline{\omega}_M^0 \\ \overline{\omega}_M^0 & \overline{\omega}_M^1 & \overline{\omega}_M^2 & \cdots & \overline{\omega}_M^{M-1} \\ \overline{\omega}_M^0 & \overline{\omega}_M^2 & \overline{\omega}_M^4 & \cdots & \overline{\omega}_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\omega}_M^0 & \overline{\omega}_M^{M-1} & \overline{\omega}_M^{2(M-1)} & \cdots & \overline{\omega}_M^{(M-1)(M-1)} \end{bmatrix}$$

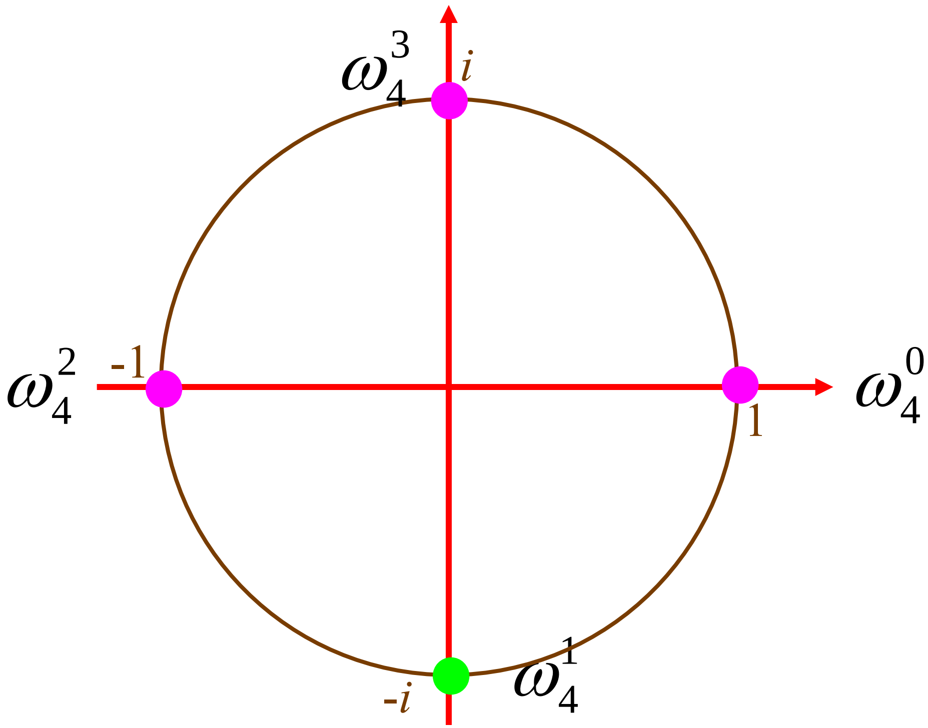
A DFT mátrixa: M=2

$$\frac{1}{2}\begin{bmatrix}\omega_2^0 & \omega_2^0\\ \omega_2^0 & \omega_2^1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$



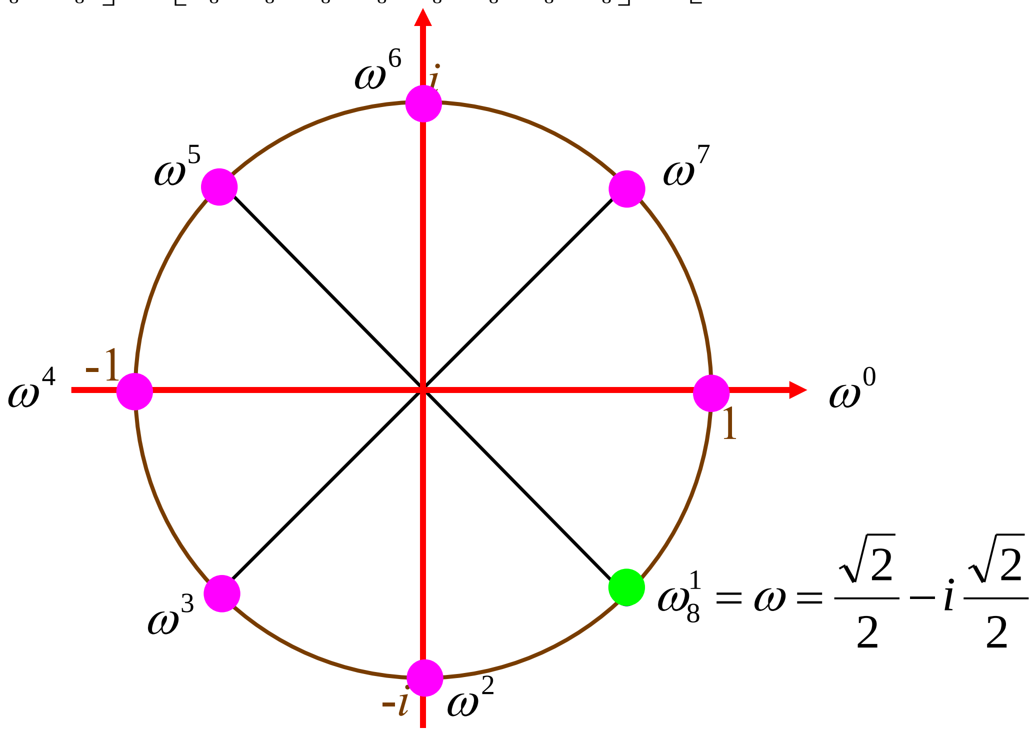
A DFT mátrixa: M=4

$$\frac{1}{4}\begin{bmatrix} \omega_4^0 & \omega_4^0 & \omega_4^0 & \omega_4^0 \\ \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ \omega_4^0 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ \omega_4^0 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \omega_4^0 & \omega_4^0 & \omega_4^0 & \omega_4^0 \\ \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ \omega_4^0 & \omega_4^2 & \omega_4^0 & \omega_4^2 \\ \omega_4^0 & \omega_4^3 & \omega_4^2 & \omega_4^1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$



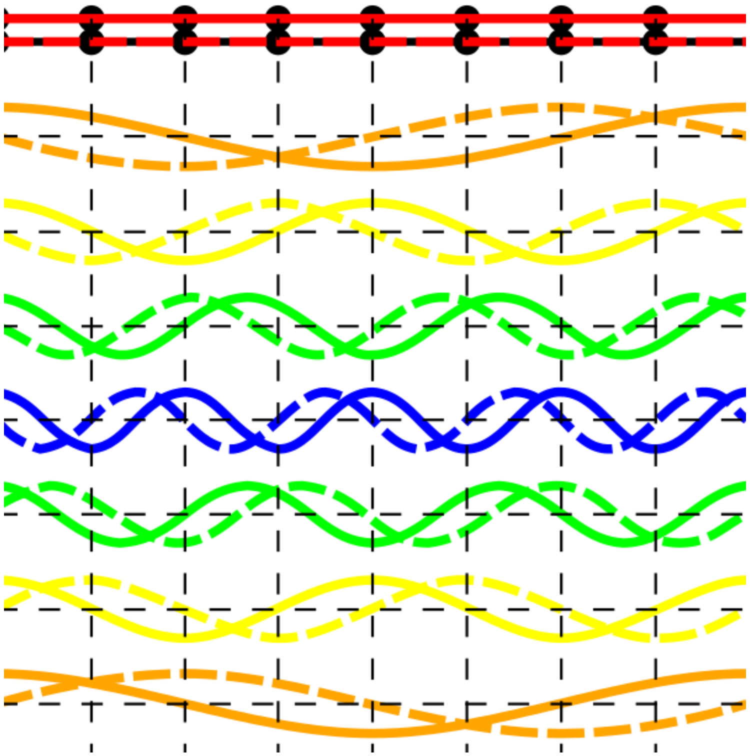
A DFT mátrixa: M=8

$$\frac{1}{8}\begin{bmatrix}
\omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 \\
\omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\
\omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^8 & \omega_8^{10} & \omega_8^{12} & \omega_8^{14} \\
\omega_8^0 & \omega_8^3 & \omega_8^6 & \omega_8^9 & \omega_8^{12} & \omega_8^{15} & \omega_8^{18} & \omega_8^{21} \\
\omega_8^0 & \omega_8^4 & \omega_8^8 & \omega_8^{12} & \omega_8^{16} & \omega_8^{20} & \omega_8^{24} & \omega_8^{28} \\
\omega_8^0 & \omega_8^5 & \omega_8^{10} & \omega_8^{15} & \omega_8^{20} & \omega_8^{25} & \omega_8^{30} & \omega_8^{35} \\
\omega_8^0 & \omega_8^6 & \omega_8^{12} & \omega_8^{18} & \omega_8^{24} & \omega_8^{30} & \omega_8^{36} & \omega_8^{42} \\
\omega_8^0 & \omega_8^7 & \omega_8^{14} & \omega_8^{21} & \omega_8^{28} & \omega_8^{35} & \omega_8^{42} & \omega_8^{49}
\end{bmatrix}
=\frac{1}{8}\begin{bmatrix}
\omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 \\
\omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\
\omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 \\
\omega_8^0 & \omega_8^3 & \omega_8^6 & \omega_8^1 & \omega_8^4 & \omega_8^7 & \omega_8^2 & \omega_8^5 \\
\omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 \\
\omega_8^0 & \omega_8^5 & \omega_8^2 & \omega_8^7 & \omega_8^4 & \omega_8^1 & \omega_8^6 & \omega_8^3 \\
\omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 & \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 \\
\omega_8^0 & \omega_8^7 & \omega_8^6 & \omega_8^5 & \omega_8^4 & \omega_8^3 & \omega_8^2 & \omega_8^1
\end{bmatrix}
=\frac{1}{8}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & -i & -i\omega & -1 & -\omega & i & i\omega \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & -i\omega & i & \omega & -1 & i\omega & -i & -\omega \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\omega & -i & i\omega & -1 & \omega & i & -i\omega \\
1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & i\omega & i & -\omega & -1 & -i\omega & -i & \omega
\end{bmatrix}$$



A DFT mátrixa: M=8

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -i & -i\omega & -1 & -\omega & i & i\omega \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\omega & i & \omega & -1 & i\omega & -i & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & -i & i\omega & -1 & \omega & i & -i\omega \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\omega & i & -\omega & -1 & -i\omega & -i & \omega \end{bmatrix}$$



$$\omega = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

valós rész: folytonos vonal
képzetes rész: szaggatott vonal

A DFT tórusz mátrixa

$$\begin{bmatrix} \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 \\ \omega_8^0 & \omega_8^5 & \omega_8^2 & \omega_8^7 & \omega_8^4 & \omega_8^1 & \omega_8^6 & \omega_8^3 \\ \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 & \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 \\ \omega_8^0 & \omega_8^7 & \omega_8^6 & \omega_8^5 & \omega_8^4 & \omega_8^3 & \omega_8^2 & \omega_8^1 \end{bmatrix}$$

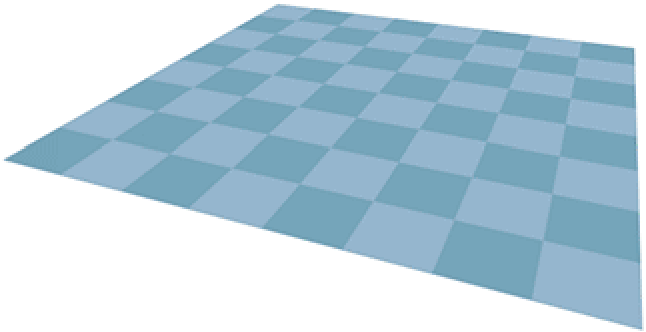
$$\begin{bmatrix} \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 \\ \omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\ \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 \\ \omega_8^0 & \omega_8^3 & \omega_8^6 & \omega_8^1 & \omega_8^4 & \omega_8^7 & \omega_8^2 & \omega_8^5 \\ \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 \\ \omega_8^0 & \omega_8^5 & \omega_8^2 & \omega_8^7 & \omega_8^4 & \omega_8^1 & \omega_8^6 & \omega_8^3 \\ \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 & \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 \\ \omega_8^0 & \omega_8^7 & \omega_8^6 & \omega_8^5 & \omega_8^4 & \omega_8^3 & \omega_8^2 & \omega_8^1 \end{bmatrix}
\begin{bmatrix} \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 \\ \omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\ \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 \\ \omega_8^0 & \omega_8^3 & \omega_8^6 & \omega_8^1 & \omega_8^4 & \omega_8^7 & \omega_8^2 & \omega_8^5 \\ \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 \\ \omega_8^0 & \omega_8^5 & \omega_8^2 & \omega_8^7 & \omega_8^4 & \omega_8^1 & \omega_8^6 & \omega_8^3 \\ \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 & \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 \\ \omega_8^0 & \omega_8^7 & \omega_8^6 & \omega_8^5 & \omega_8^4 & \omega_8^3 & \omega_8^2 & \omega_8^1 \end{bmatrix}
\begin{bmatrix} \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 \\ \omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\ \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 \\ \omega_8^0 & \omega_8^3 & \omega_8^6 & \omega_8^1 & \omega_8^4 & \omega_8^7 & \omega_8^2 & \omega_8^5 \\ \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 \\ \omega_8^0 & \omega_8^5 & \omega_8^2 & \omega_8^7 & \omega_8^4 & \omega_8^1 & \omega_8^6 & \omega_8^3 \\ \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 & \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 \\ \omega_8^0 & \omega_8^7 & \omega_8^6 & \omega_8^5 & \omega_8^4 & \omega_8^3 & \omega_8^2 & \omega_8^1 \end{bmatrix}$$

$$\begin{bmatrix} \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 \\ \omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\ \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 \\ \omega_8^0 & \omega_8^3 & \omega_8^6 & \omega_8^1 & \omega_8^4 & \omega_8^7 & \omega_8^2 & \omega_8^5 \end{bmatrix}$$

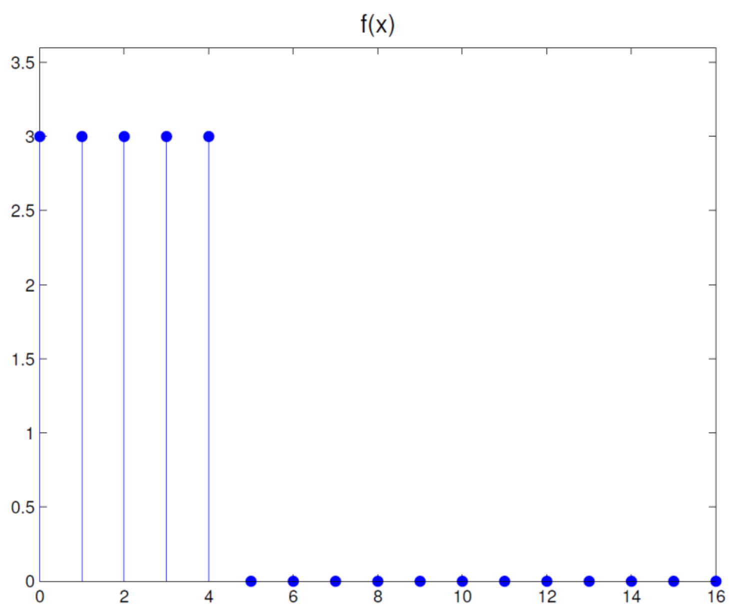
A DFT tórusz mátrixa

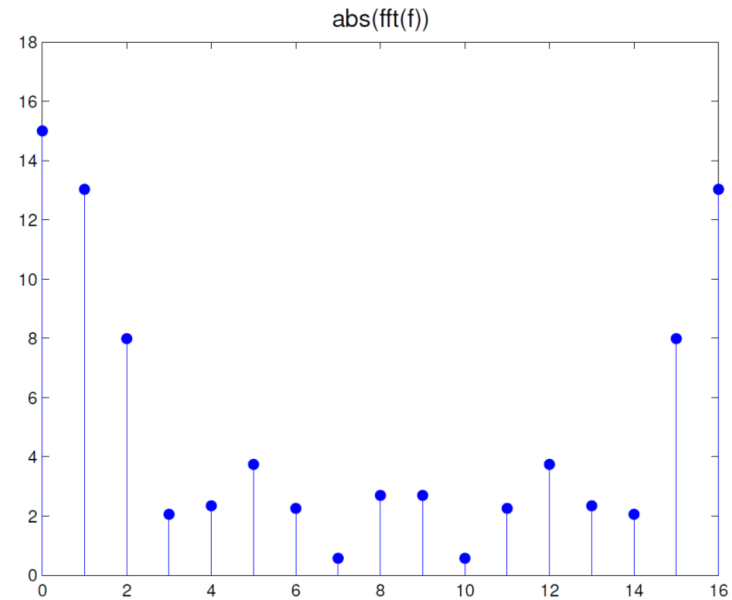
$$\begin{bmatrix}
\omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 & \omega_8^0 \\
\omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\
\omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^0 & \omega_8^2 & \omega_8^4 & \omega_8^6 \\
\omega_8^0 & \omega_8^3 & \omega_8^6 & \omega_8^1 & \omega_8^4 & \omega_8^7 & \omega_8^2 & \omega_8^5 \\
\omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 & \omega_8^0 & \omega_8^4 \\
\omega_8^0 & \omega_8^5 & \omega_8^2 & \omega_8^7 & \omega_8^4 & \omega_8^1 & \omega_8^6 & \omega_8^3 \\
\omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 & \omega_8^0 & \omega_8^6 & \omega_8^4 & \omega_8^2 \\
\omega_8^0 & \omega_8^7 & \omega_8^6 & \omega_8^5 & \omega_8^4 & \omega_8^3 & \omega_8^2 & \omega_8^1
\end{bmatrix}$$

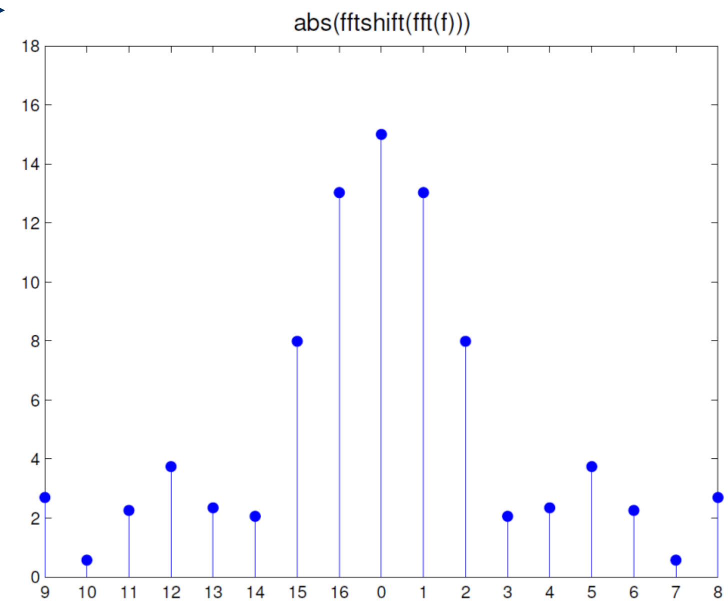
$$M = 8$$



Példák 1D DFT-re

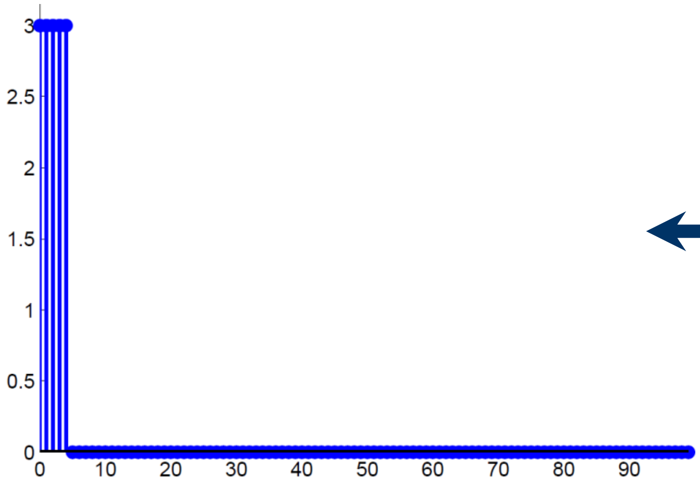


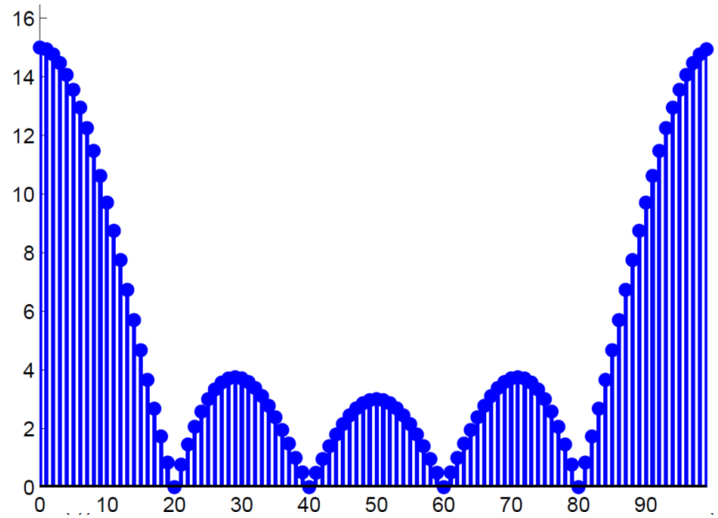




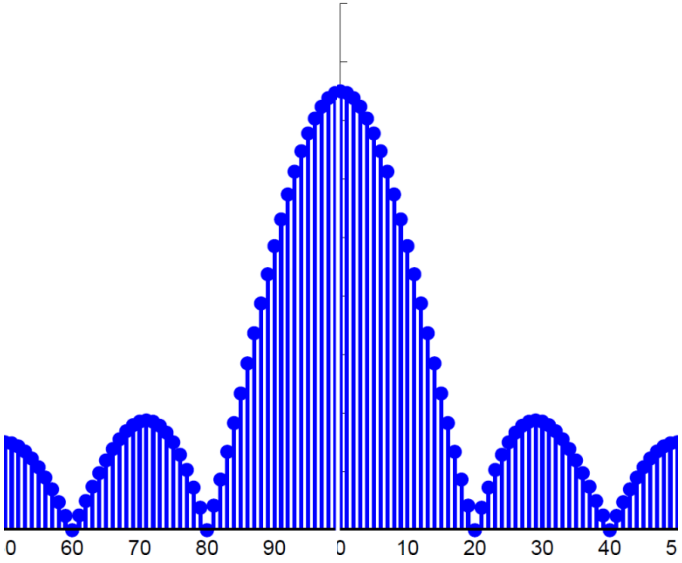
középre tolt origó

Példák 1D DFT-re

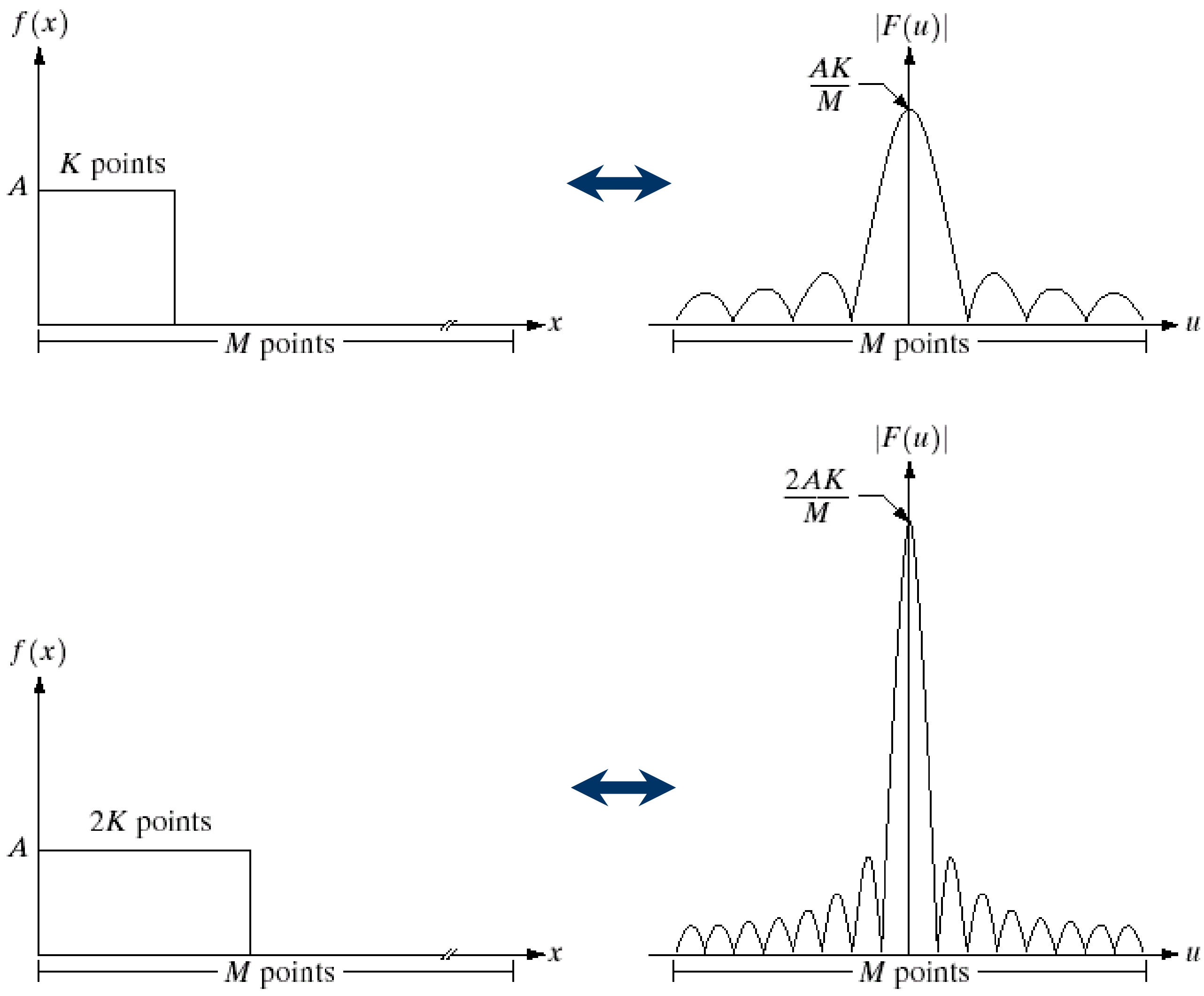




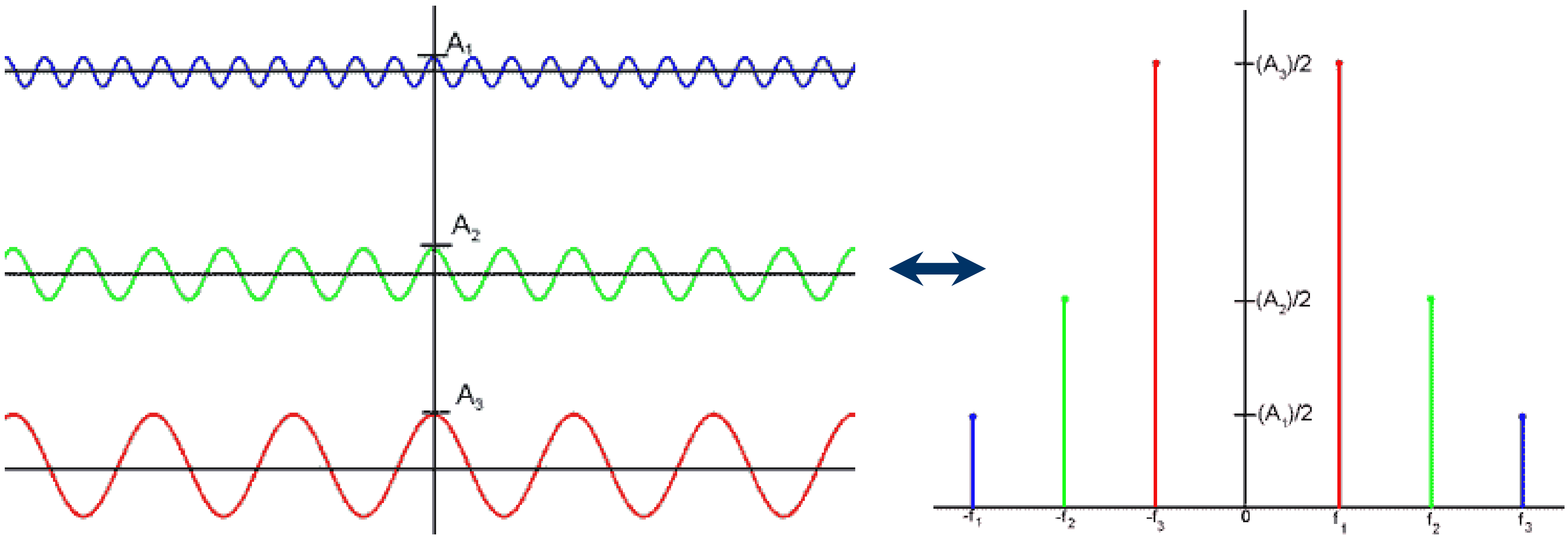
középre tolt origó



Példák 1D DFT-re



Példa 1D DFT-re



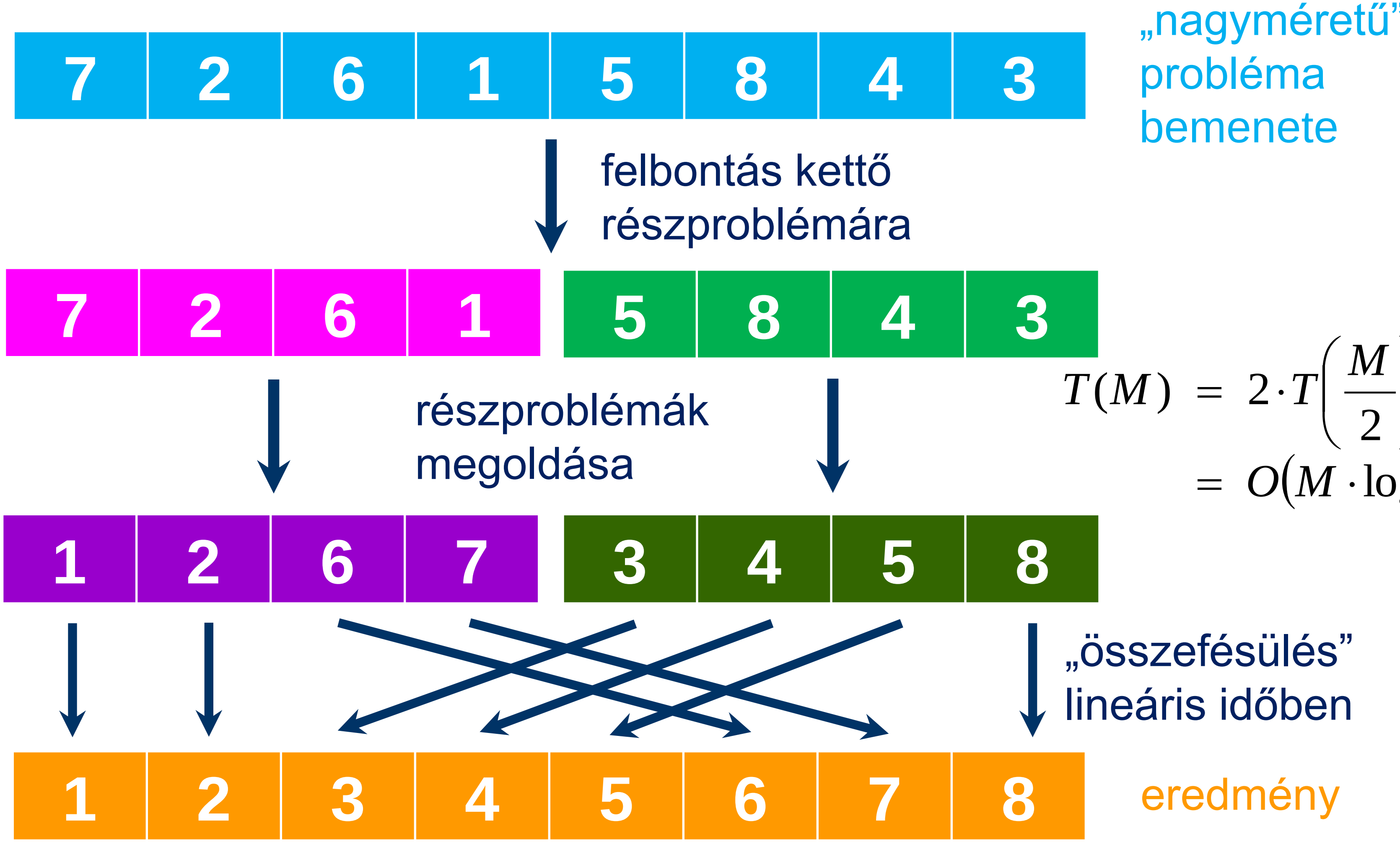
Az 1D DFT műveletigénye

$$\frac{1}{M}\begin{bmatrix} \omega_M^0 & \omega_M^0 & \omega_M^0 & \cdots & \omega_M^0 \\ \omega_M^0 & \omega_M^1 & \omega_M^2 & \cdots & \omega_M^{M-1} \\ \omega_M^0 & \omega_M^2 & \omega_M^4 & \cdots & \omega_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_M^0 & \omega_M^{M-1} & \omega_M^{2(M-1)} & \cdots & \omega_M^{(M-1)(M-1)} \end{bmatrix} \cdot \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix} = \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(M-1) \end{bmatrix}$$

$M \times M$ $\qquad$ $M \times 1$ $\qquad$ $M \times 1$

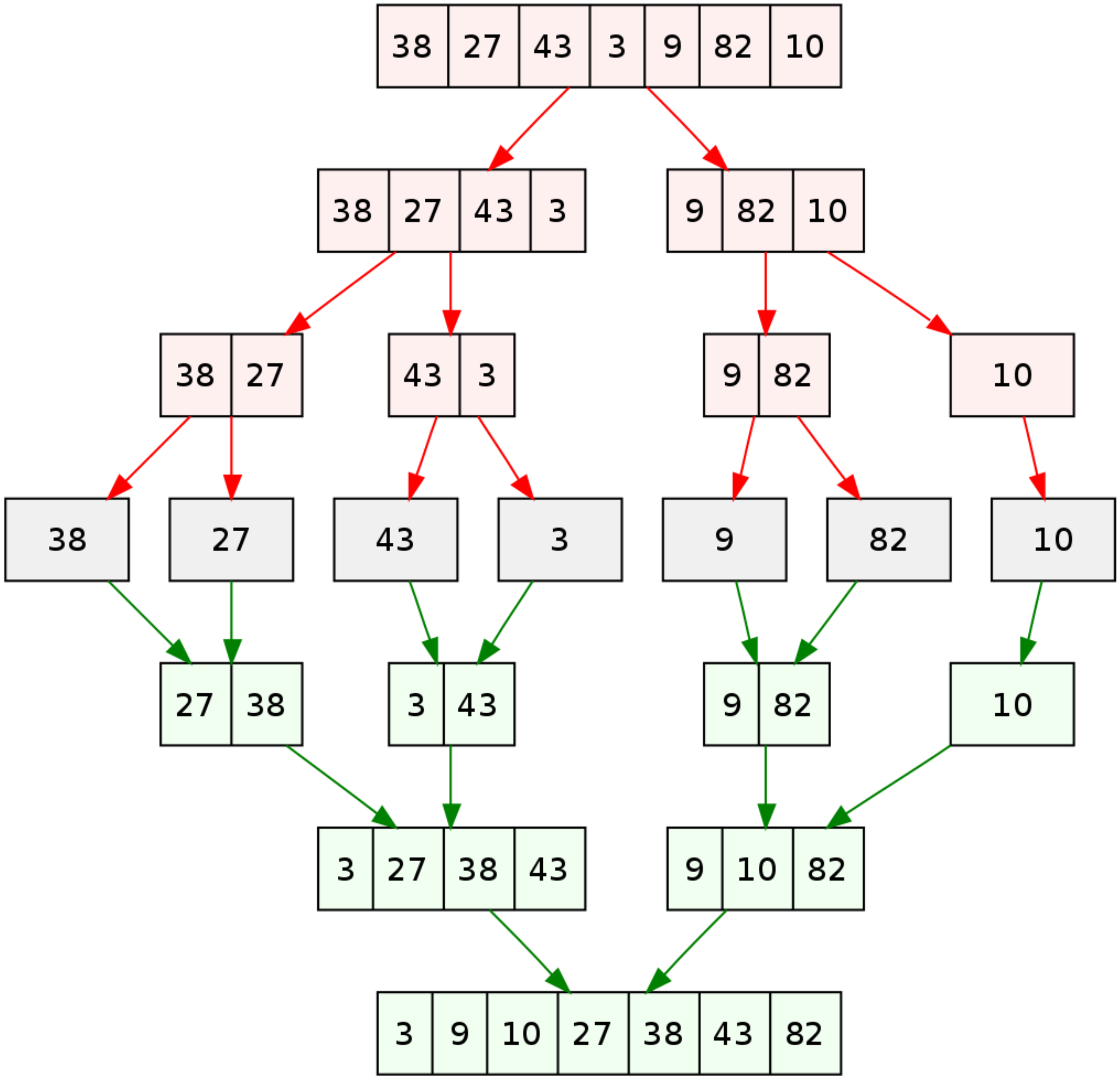
$$M \cdot M = M^2 \text{ szorzás}$$

Merge sort



$$
\begin{aligned}
T(M) &= 2\cdot T\left(\frac{M}{2}\right)+O(M)\\
&= O(M\cdot\log_2 M)
\end{aligned}
$$

Merge sort – divide-and-conquer



$$T(M) = 2 \cdot T\left(\frac{M}{2}\right) + O(M)$$

$$= O(M \cdot \log_2 M)$$

Gyors Fourier transzformáció
(Fast Fourier Transform, FFT)

A Fourier transzformáció MxM-es mátrixa speciális (szimmetrikus és csak M-féle elemet tartalmaz), ugyanis:

$$A(u,v) = \frac{1}{M} \cdot \omega_M^{uv} \qquad (u,v = 0,1,\ldots,M-1).$$

Ezt kihasználva, egy M-elemű jel transzformációja megkapható kettő ($M/2$)-elemű jel transzformáltjának $O(M)$ (lineáris) idejű „összefésülésével”, így az időigény $O(M \cdot \log_2 M)$.

Cooley, Tukey, 1965

FFT

$$M = 2^n = 2K$$

$$\omega_M = e^{-2\pi i/M}$$

páros

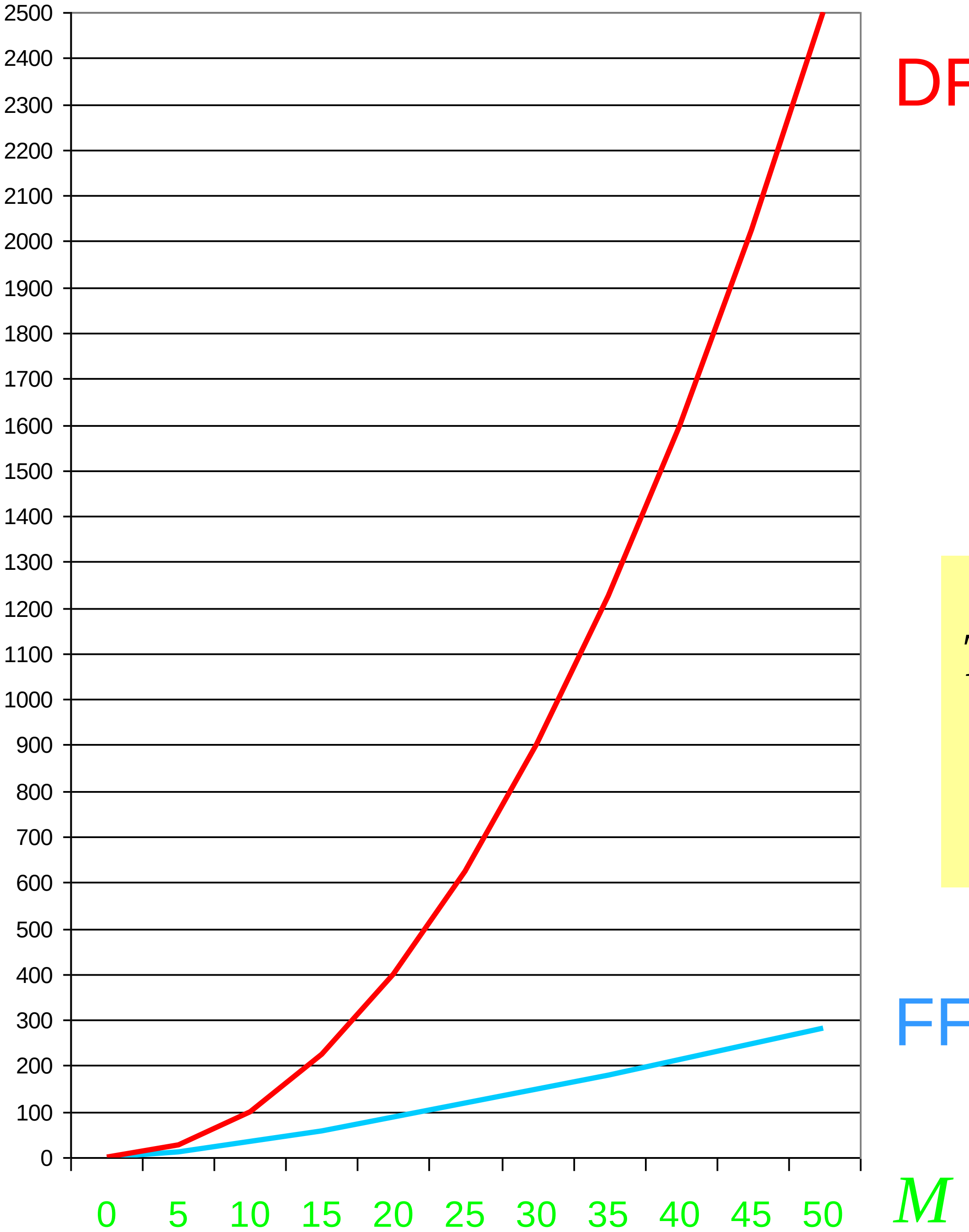
páratlan

$$F(X) = \frac{1}{2K}\sum_{x=0}^{2K-1} f(x)\cdot\omega_{2K}^{xX}$$

$$= \frac{1}{2}\left[\frac{1}{K}\sum_{x=0}^{K-1} f(2x)\cdot\omega_{2K}^{2xX} + \frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)\cdot\omega_{2K}^{(2x+1)X}\right]$$

$$= \frac{1}{2}\left[\frac{1}{K}\sum_{x=0}^{K-1} f(2x)\cdot\omega_{K}^{xX} + \frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)\cdot\omega_{K}^{xX}\cdot\omega_{2K}^{X}\right]$$

$$= \frac{1}{2}\left[F_{even}(X) + \omega_{2K}^{X}\cdot F_{odd}(X)\right]$$

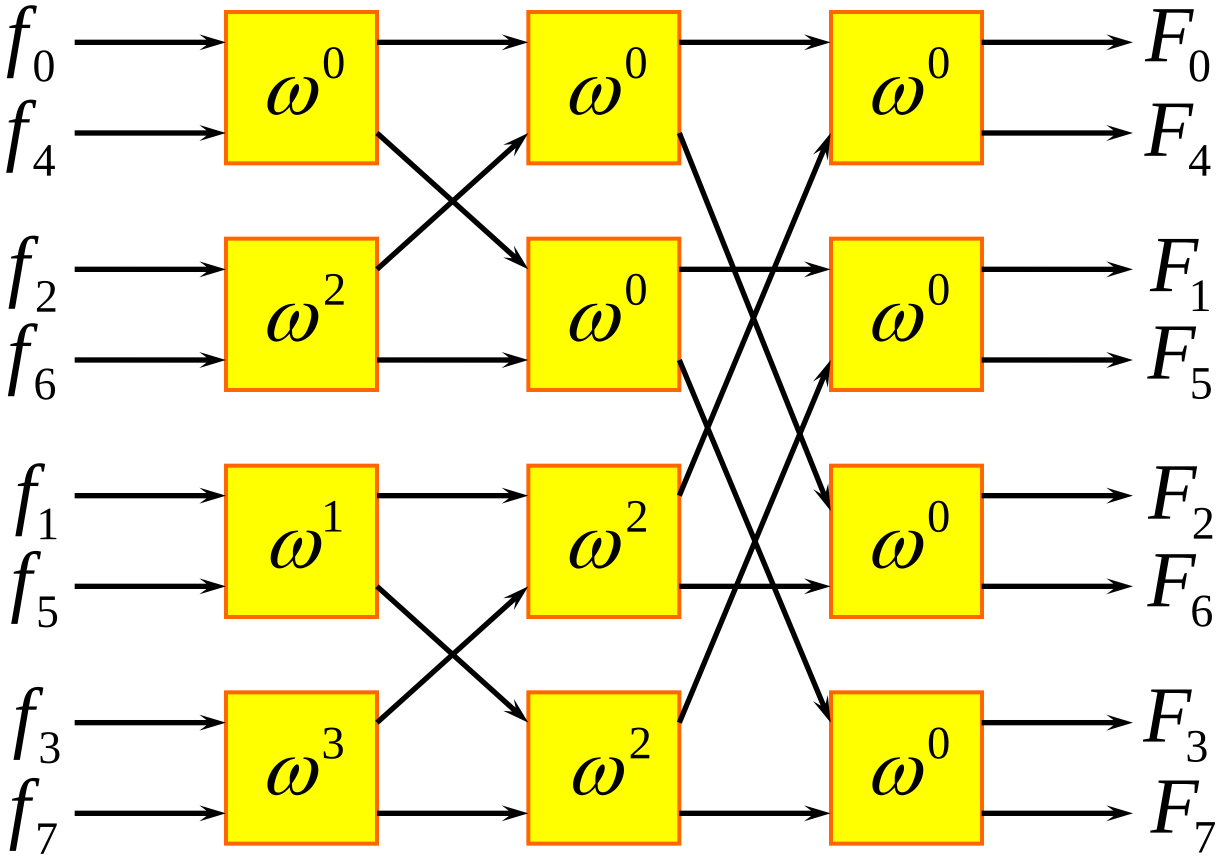


A DFT és az FFT időigényének összevetése

$$T(M) = 2 \cdot T\left(\frac{M}{2}\right) + O(M)$$

$$= O\left(M \cdot \log_2 M\right)$$

Párhuzamos FFT hálózat

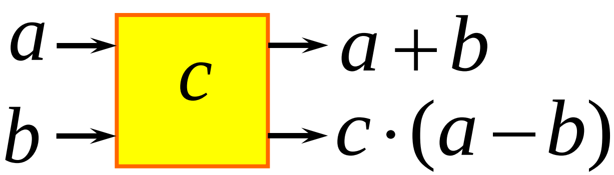


M=8 elemre

$$f_i = f(i)$$

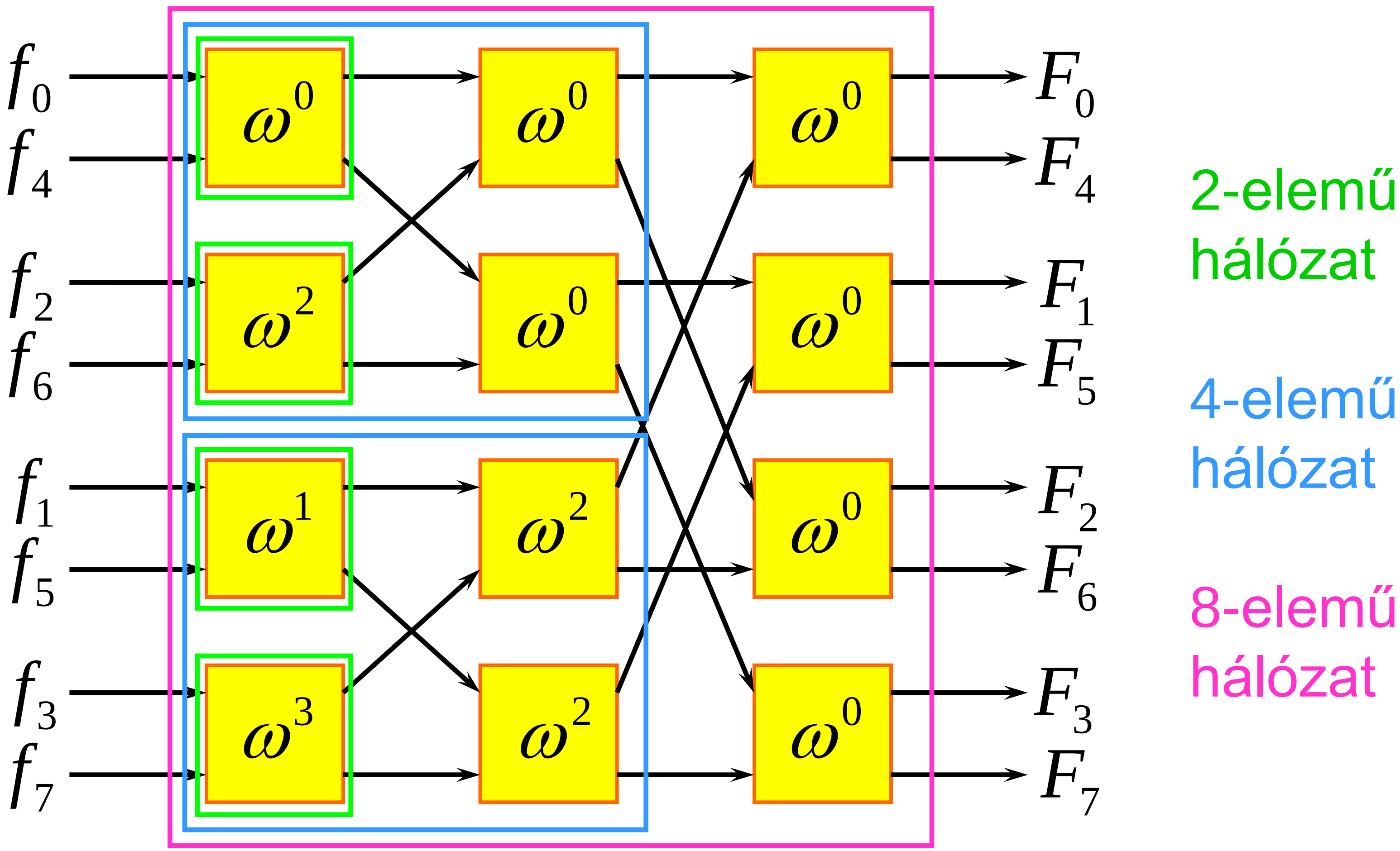
$$F_i = F(i)$$

$$(i = 0, \ldots, M-1)$$

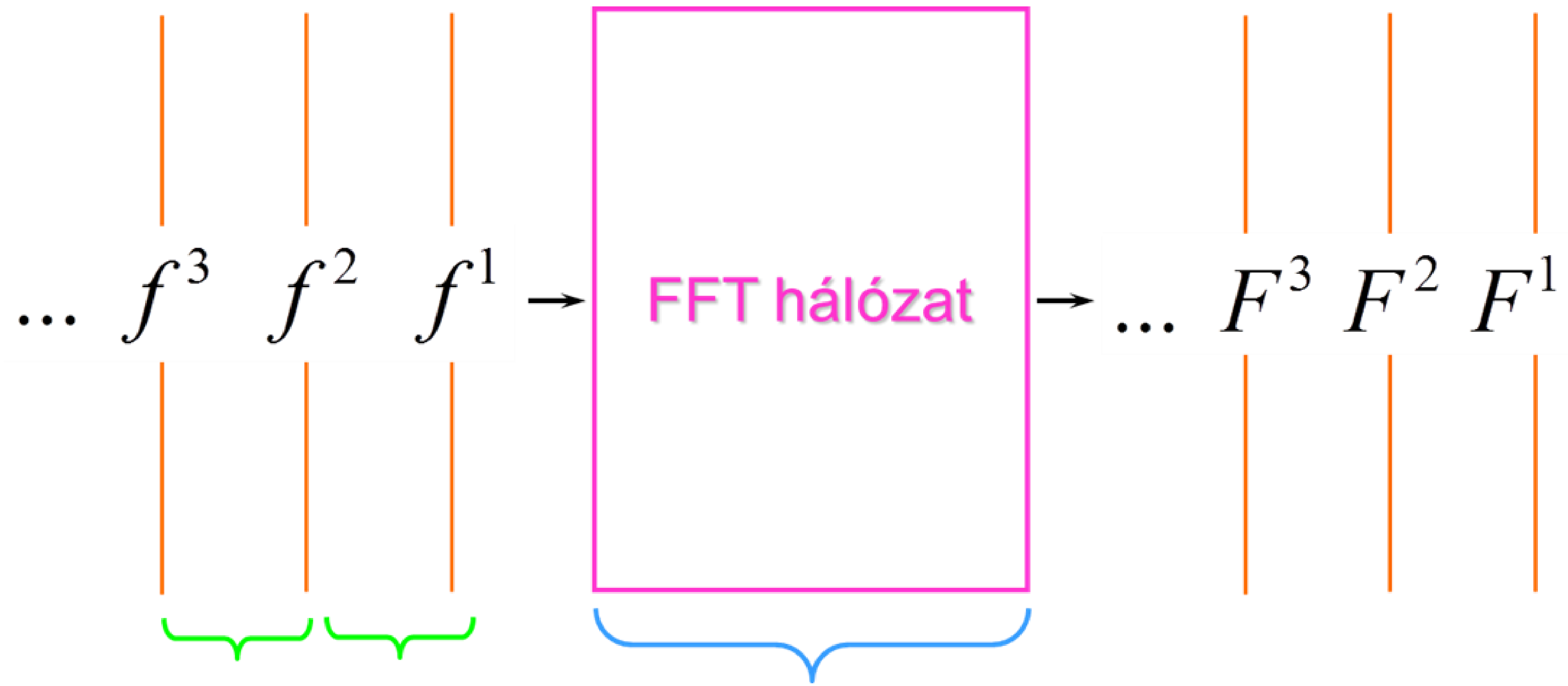


az $(M/2)\cdot\log_2 M$ uniform egységből álló hálózat egy elemének funkciója

Párhuzamos FFT hálózat



Párhuzamos FFT hálózat



a mintavételezési/
követési idő: Δt

egy transzformáció
végrehajtási ideje: $\log_2 M \cdot \Delta t$

2D Fourier-transzformáció

DFT

$$F(X,Y) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\cdot w(x,y,X,Y)$$

$$w(x,y,X,Y) = \frac{1}{MN} e^{-2\pi i\left(\frac{xX}{M}+\frac{yY}{N}\right)}$$

IDFT

$$f(x,y) = \sum_{X=0}^{M-1}\sum_{Y=0}^{N-1} F(X,Y)\cdot w'(x,y,X,Y)$$

$$w'(x,y,X,Y) = e^{2\pi i\left(\frac{xX}{M}+\frac{yY}{N}\right)}$$

$$(\ x, X = 0,1,\ldots, M-1; \quad y, Y = 0,1,\ldots, N-1\ )$$

2D Fourier-transzformáció

$$f(x,y) = \sum_{X=0}^{M-1}\sum_{Y=0}^{N-1} F(X,Y)\cdot w'(x,y,X,Y)$$

egy pixel

bázisfüggvény

súly

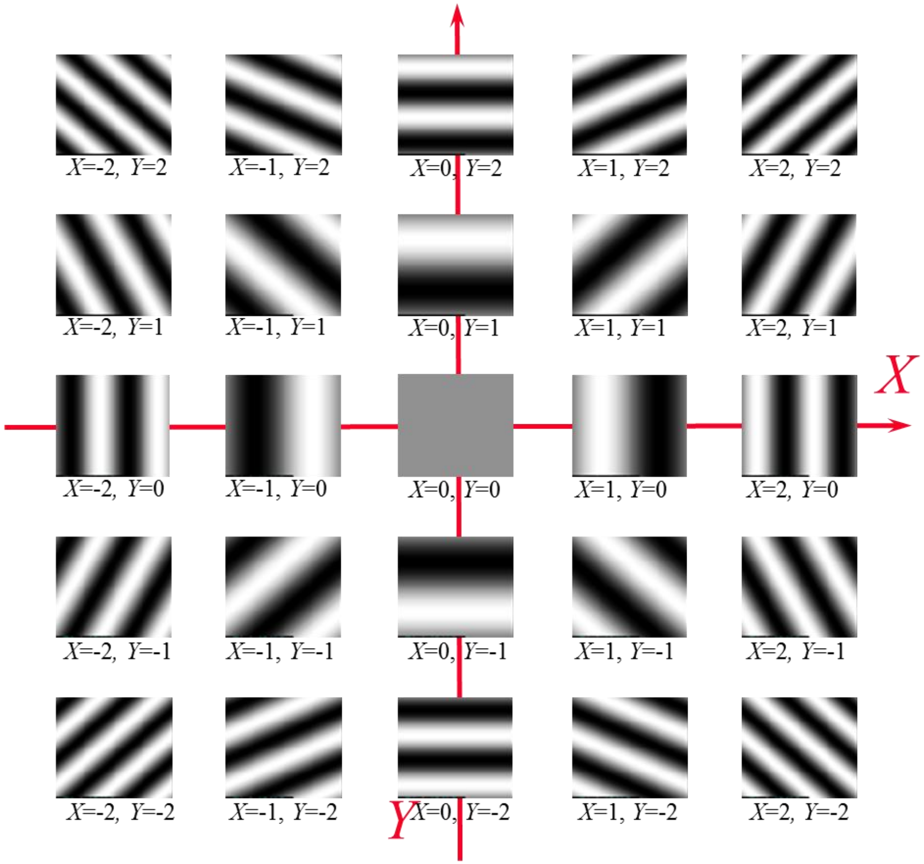
$$f = \sum_{X=0}^{M-1}\sum_{Y=0}^{N-1} F(X,Y)\cdot W'(X,Y)$$

teljes kép

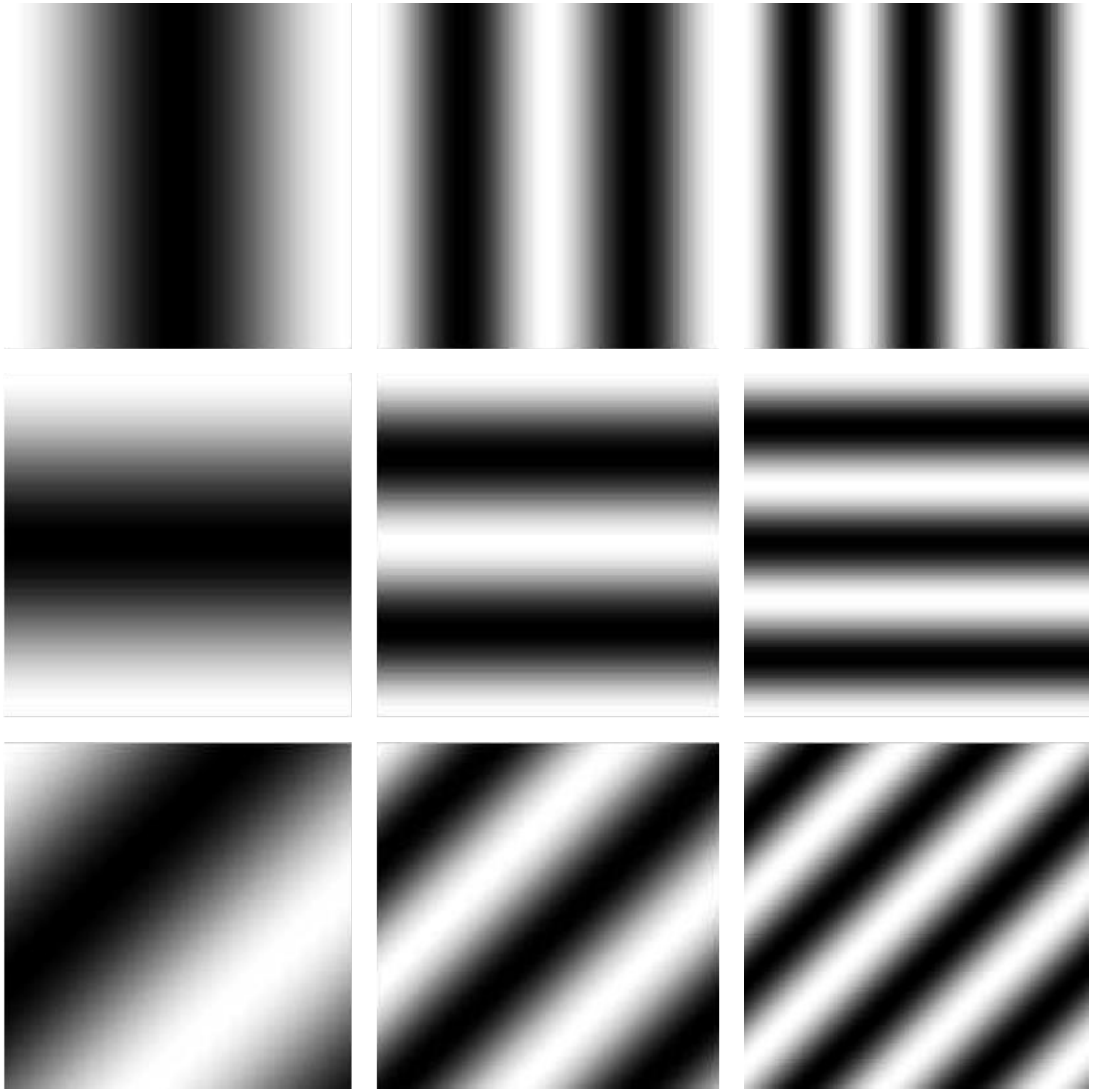
báziskép

2D DFT bázisképei

$$\mathrm{Re}\left(e^{-2\pi i\left(\frac{xX}{M}+\frac{yY}{N}\right)}\right)=\cos\left(\frac{2\pi}{M}xX+\frac{2\pi}{N}yY\right)$$

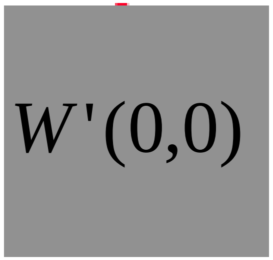


2D DFT bázisképei

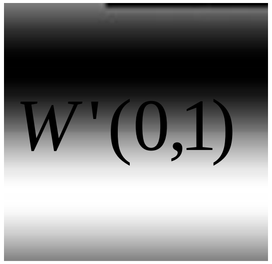


2D Fourier-transzformáció

$$f(x,y) = F(0,0) \cdot W'(0,0) +$$



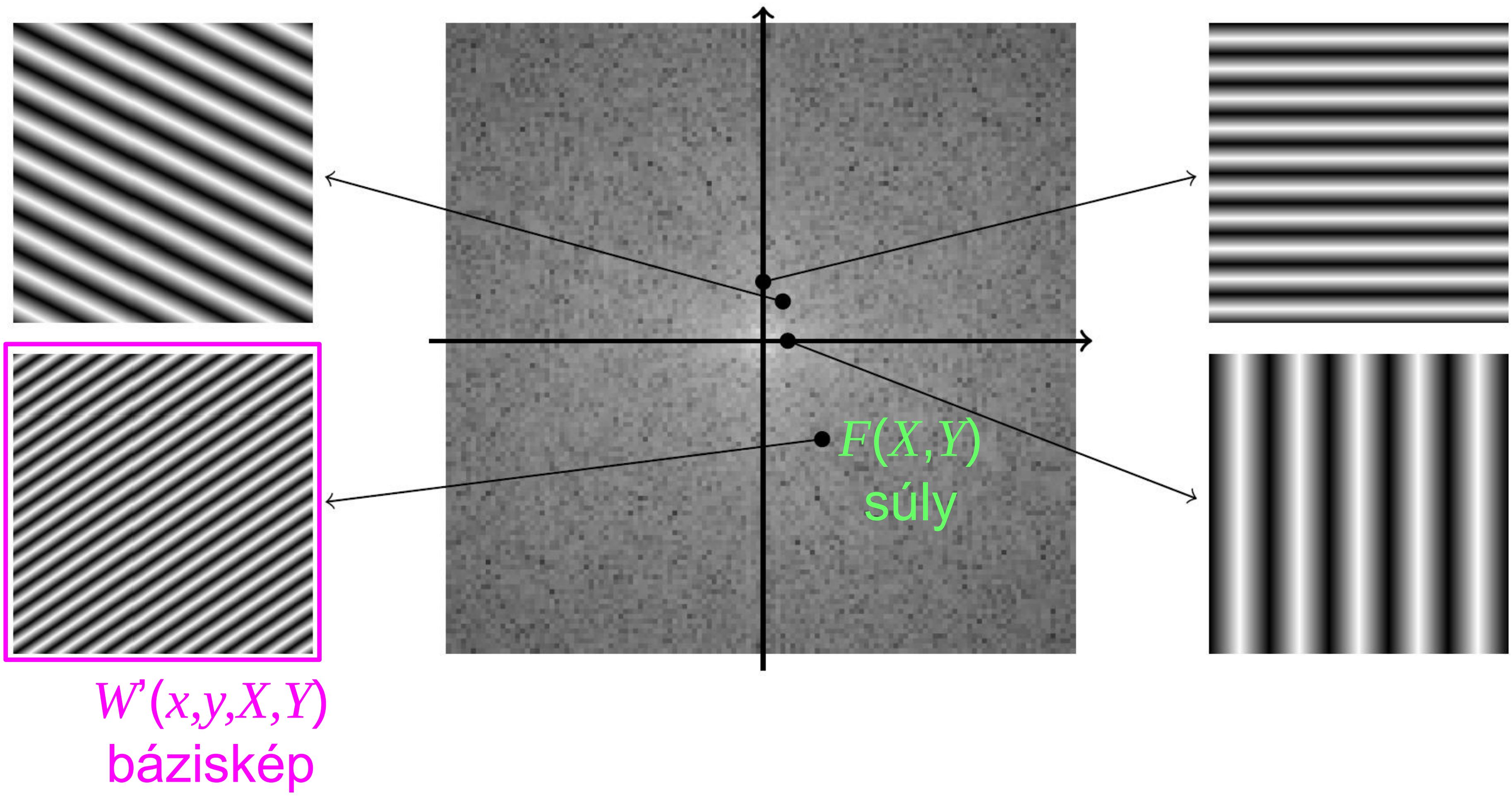
$$F(0,1) \cdot W'(0,1) +$$



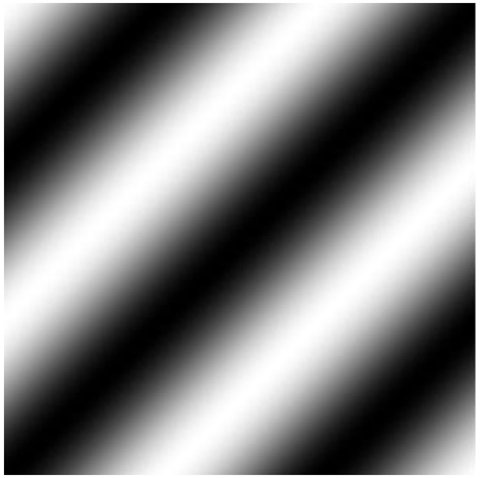
$$F(0,2) \cdot W'(0,2) +$$

$$\ldots$$

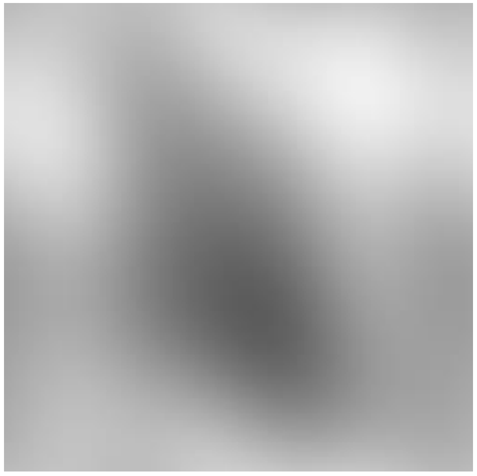
2D Fourier-transzformáció



Bázisképek lineáris kombinációja




kettő a 2^{16}-féle
256x256-os
báziskép közül



lineáris kombinációjuk
(súlyozott összegük)

Bázisképek lineáris kombinációja

80

300

2000

$2^{16} = 65\,536$

... báziskép súlyozott összege

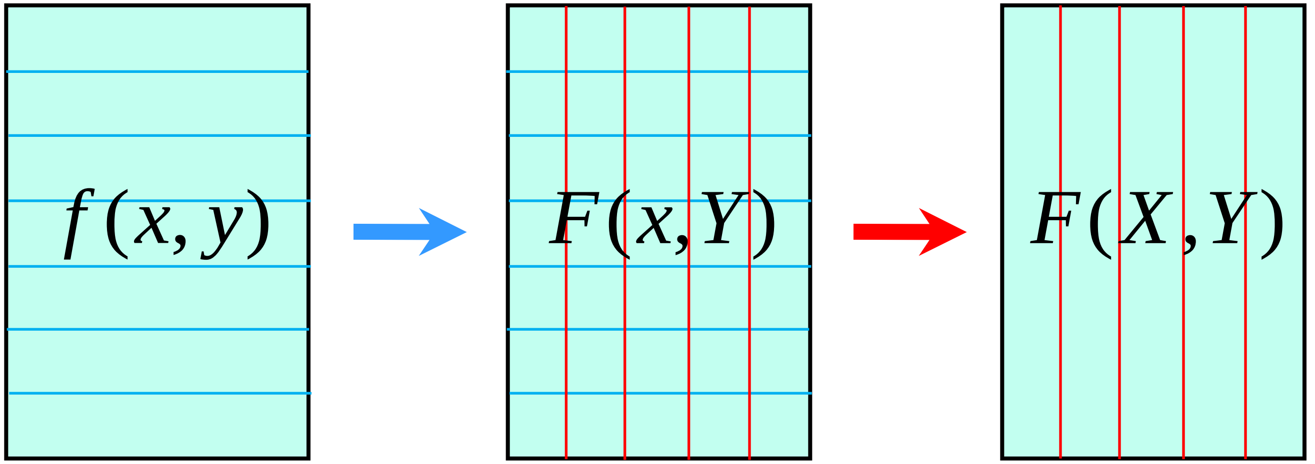
A 2D DFT szeparálható

$$F(X,Y)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\cdot e^{-i2\pi(xX/M+yY/N)}$$

$$=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\cdot e^{-i2\pi yY/N}\cdot e^{-i2\pi xX/M}$$

$$=\frac{1}{M}\sum_{x=0}^{M-1}\underbrace{\left[\frac{1}{N}\sum_{y=0}^{N-1}f(x,y)\cdot e^{-i2\pi yY/N}\right]}_{F(x,Y)=}e^{-i2\pi xX/M}$$

A 2D DFT szeparálható

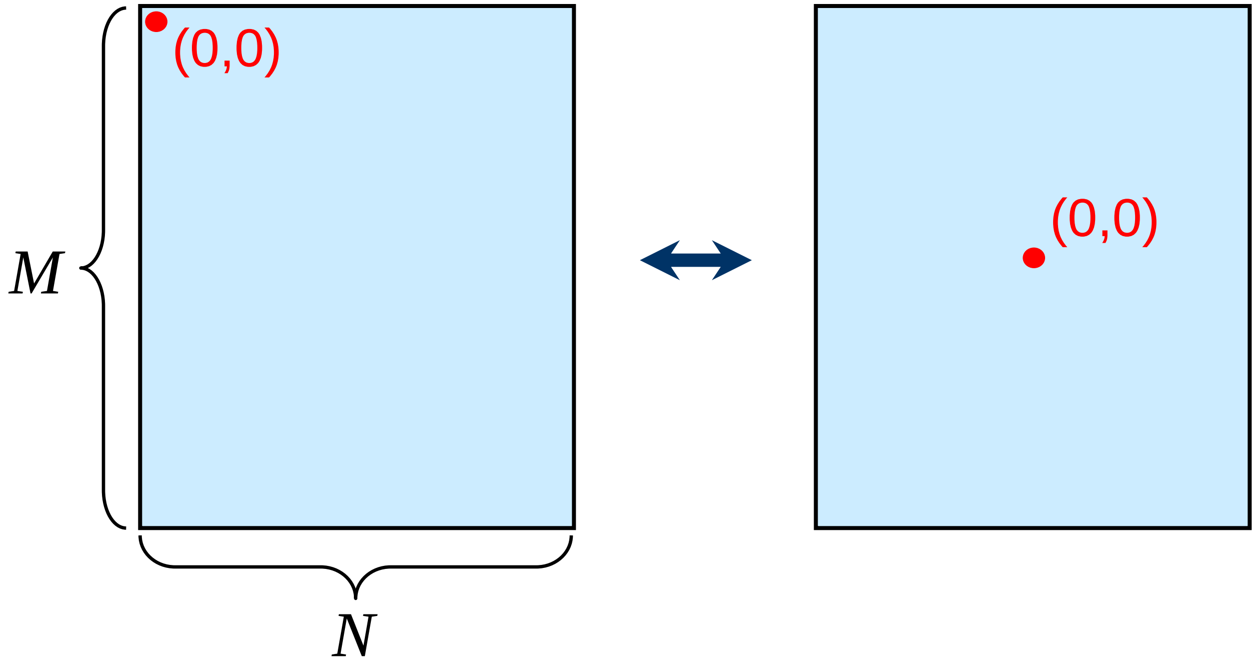


sorok transzformációja

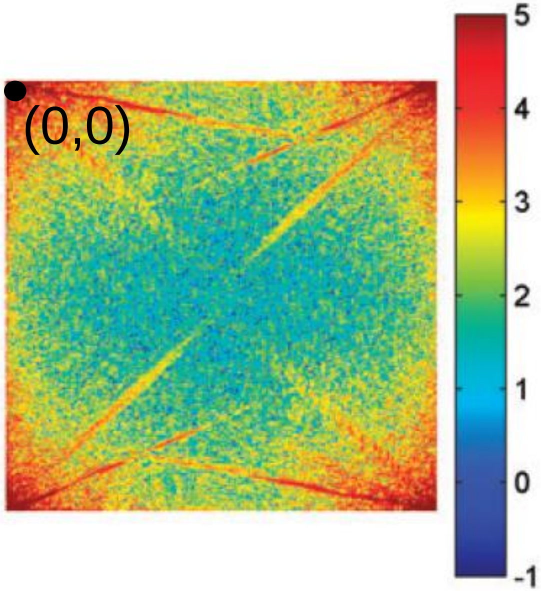
oszlopok transzformációja

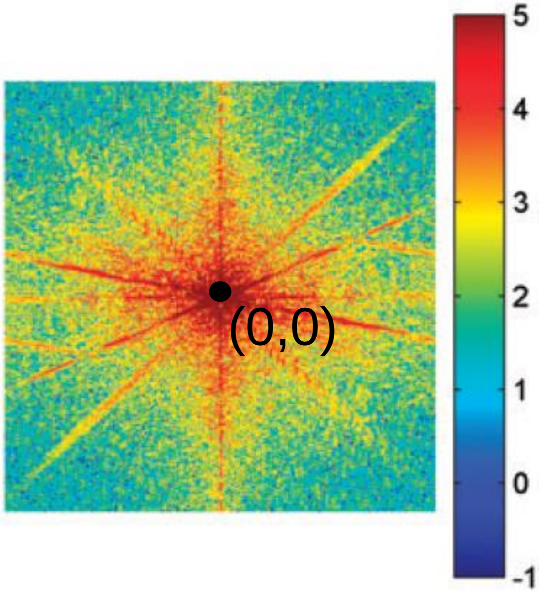
Az origó középre tolása

$$(-1)^{x+y} f(x,y) \leftrightarrow F(X-\frac{M}{2}, Y-\frac{N}{2})$$

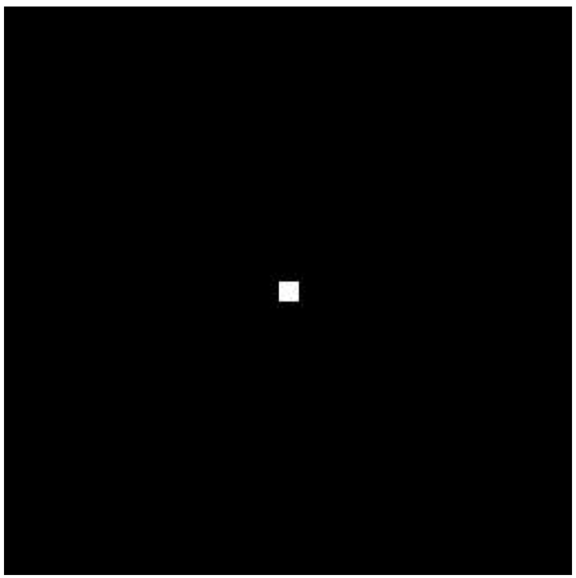


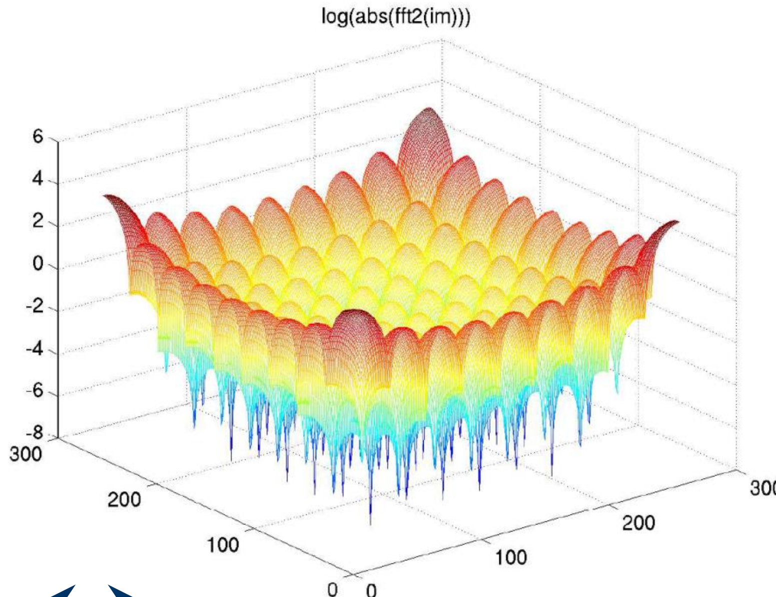
Az origó középre tolása

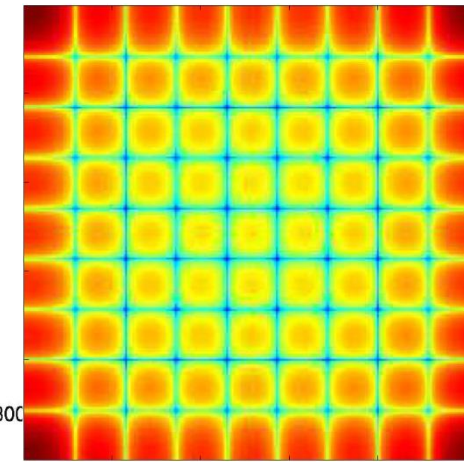


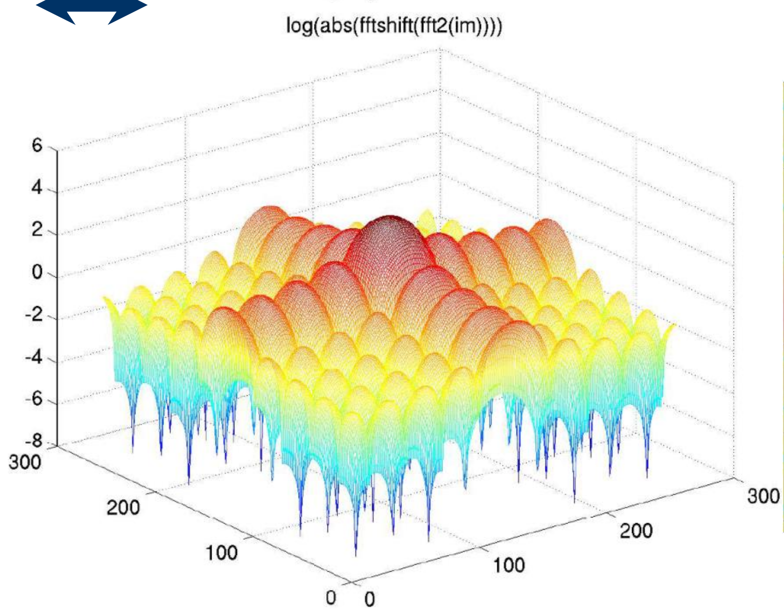


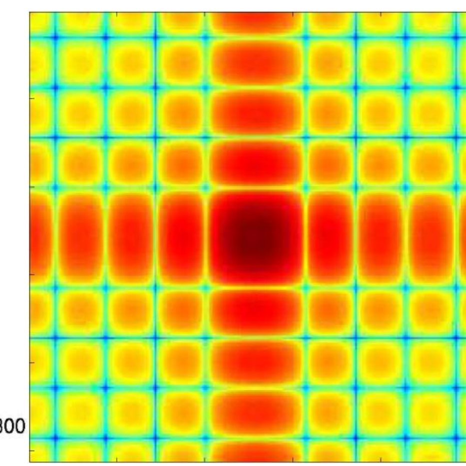
Az origó középre tolása



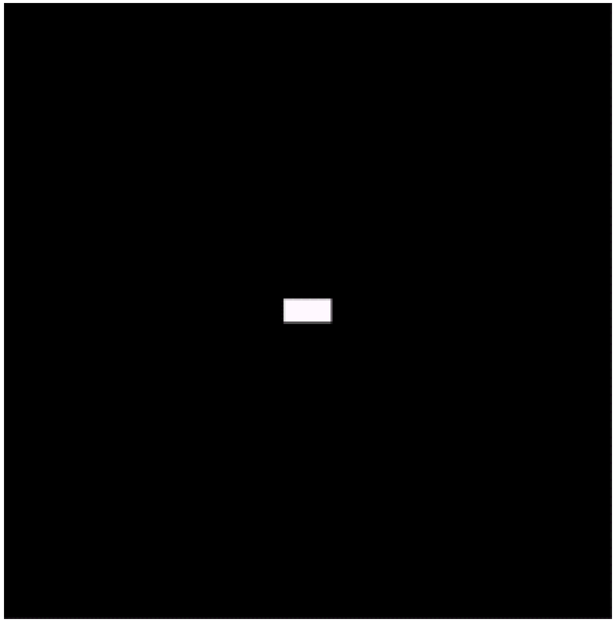




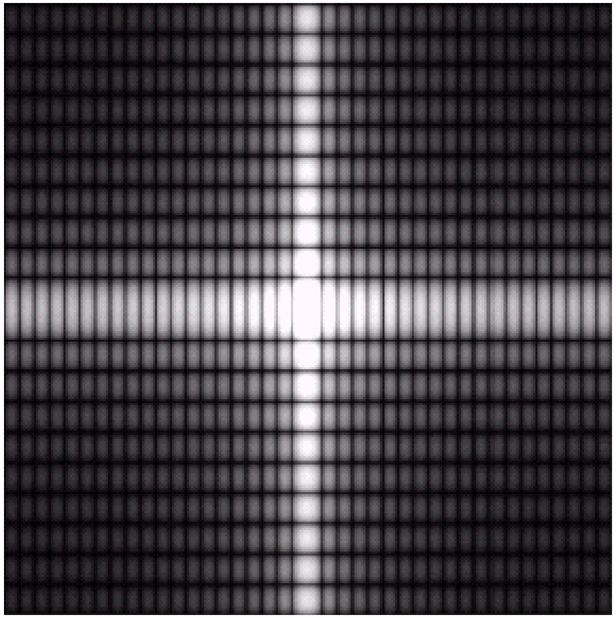




Kép-tér és frekvencia-tér



$f(x)$

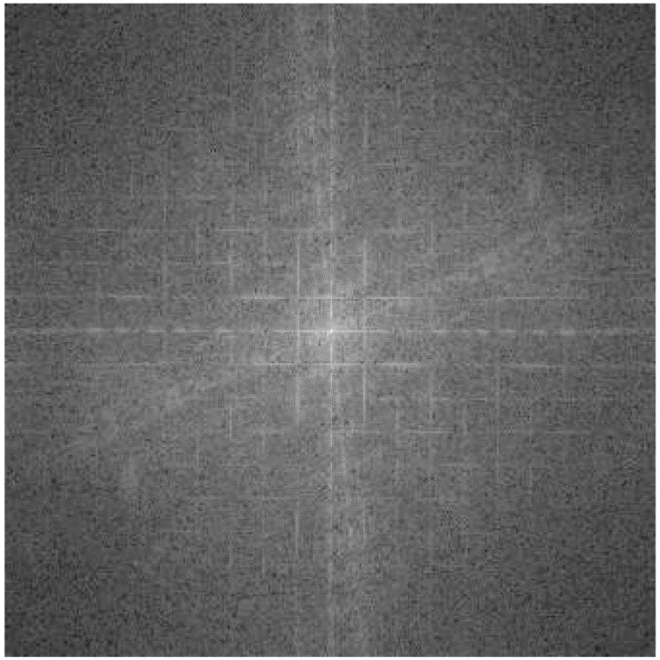


$|F(X,Y)|$

Fourier-pár

Kép-tér és frekvencia-tér

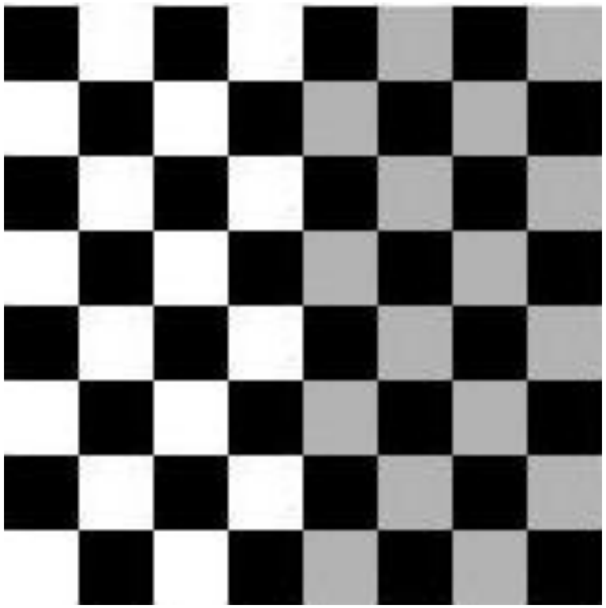
$\leftrightarrow$



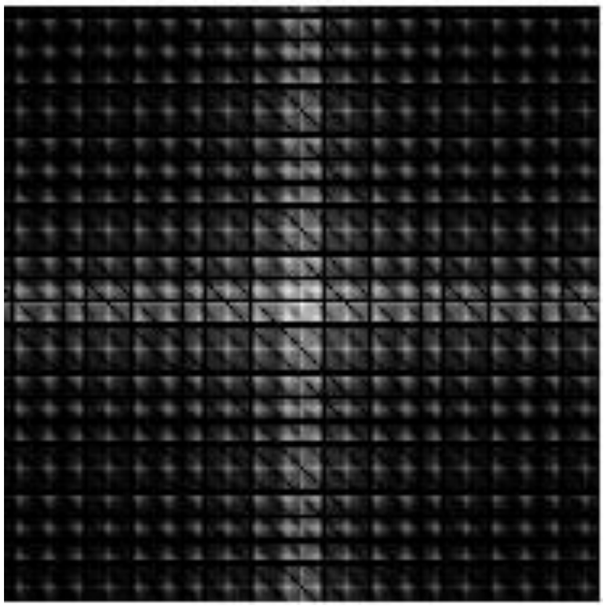
$f(x)$ $|F(X,Y)|$

Fourier-pár

Kép-tér és frekvencia-tér



$\leftrightarrow$

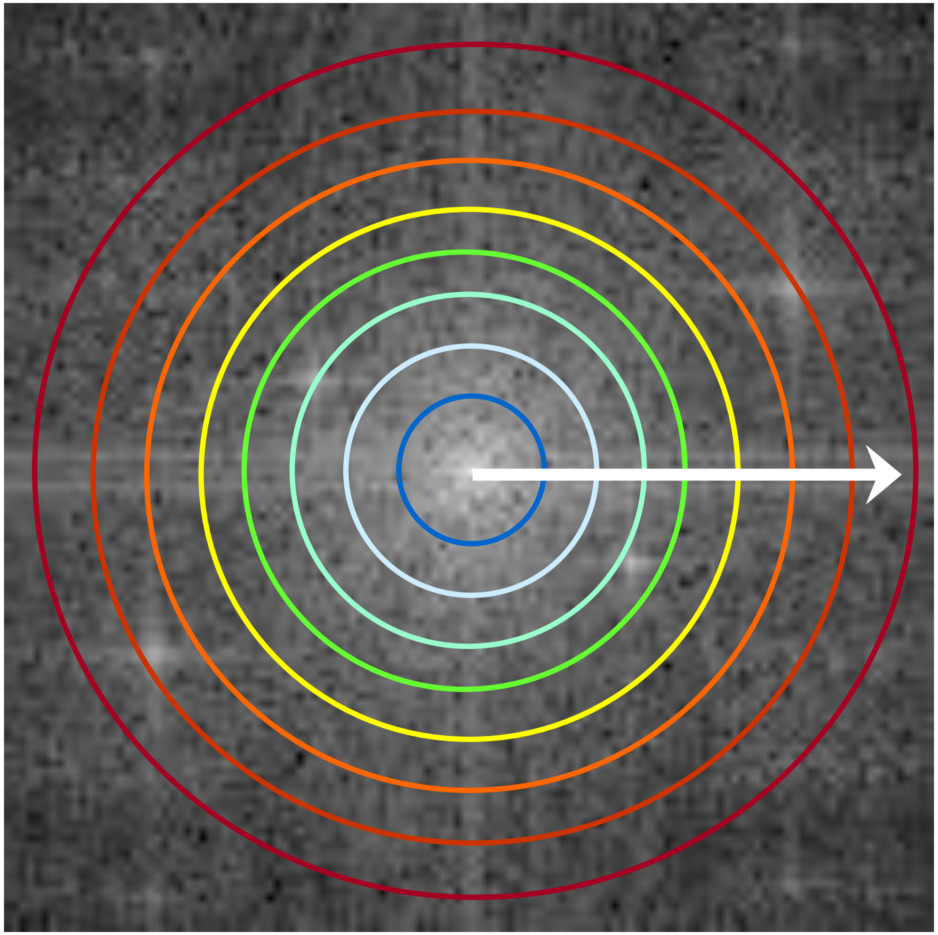


$f(x)$

$|F(X,Y)|$

Fourier-pár

Frekvencia tér



alacsony frekvencia

magas frekvencia

$$\left|F(X,Y)\right|$$

A frekvenciakép megjelenítése

Általában a Fourier-spektrum az origótól távolodva gyorsan lecseng, ezért az

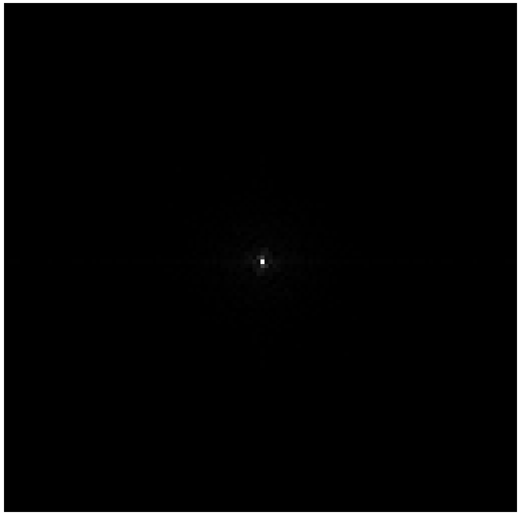
$$|F(x,y)|$$

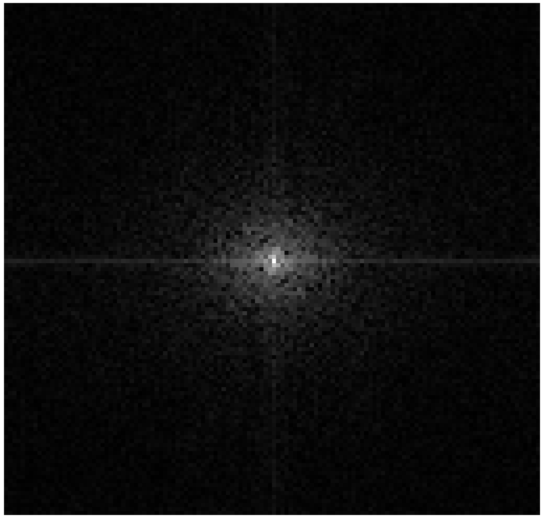
helyett a

$$\log(\ 1 + |F(X,Y)|\ )$$

megjelenítése javallott.

A frekvenciakép megjelenítése

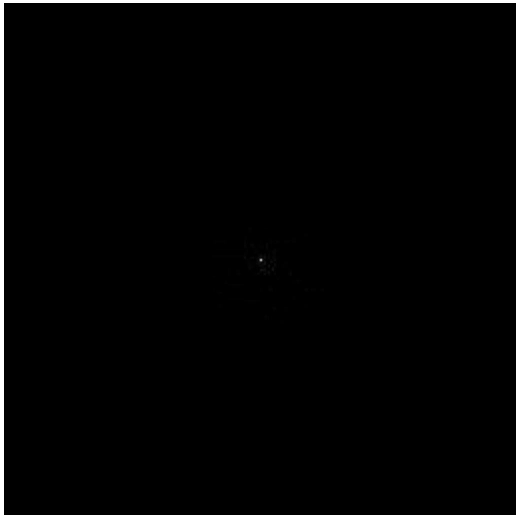




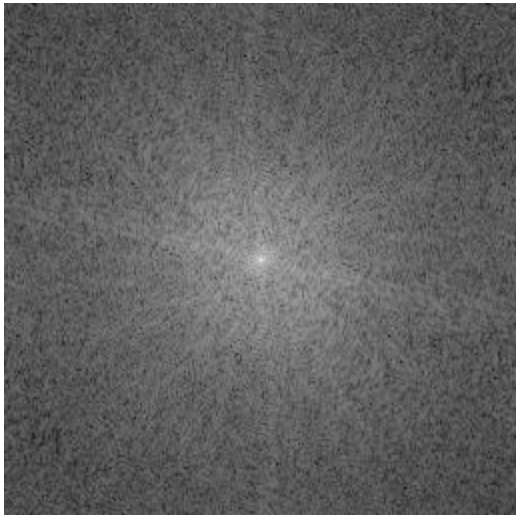
$f(x)$ $|F(X,Y)|$ $\log(1+|F(X,Y)|)$

A frekvenciakép megjelenítése

$f(x)$



$|F(X,Y)|$

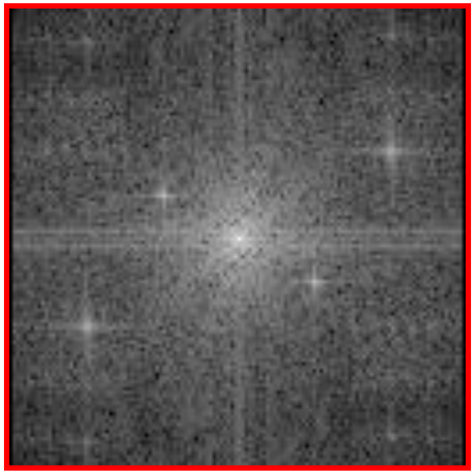


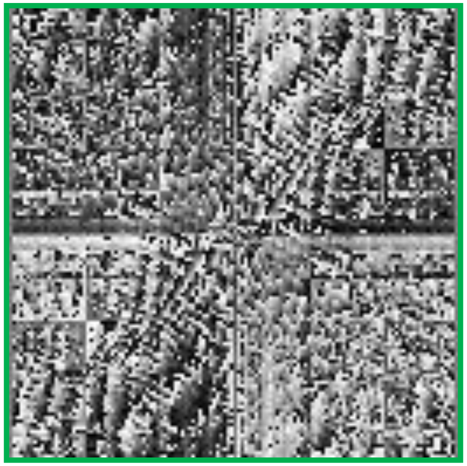
$\log(1+|F(X,Y)|)$

A frekvenciakép megjelenítése

$f(x, y)$

$\leftrightarrow$

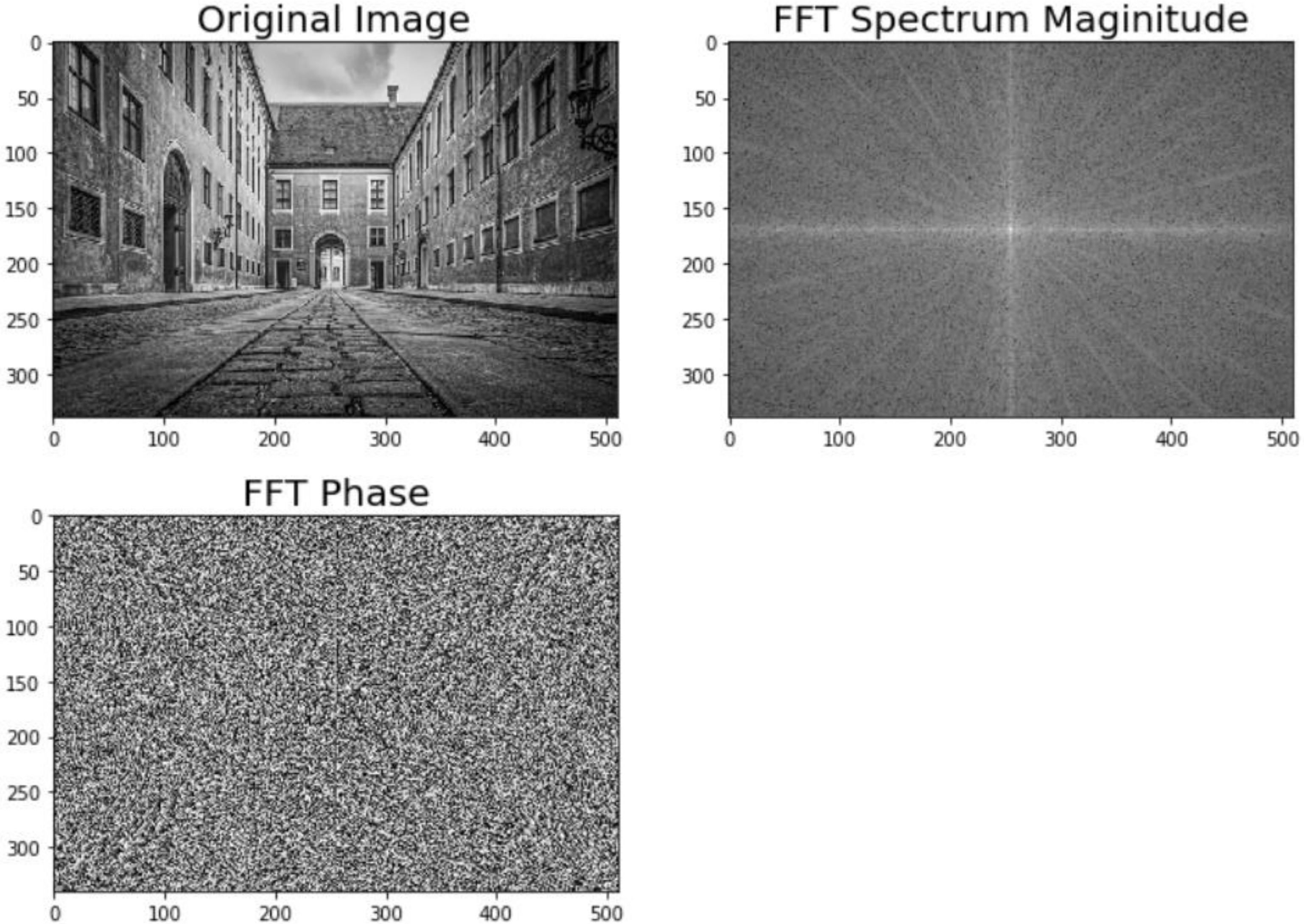




magnitúdó: $|F(X,Y)| = \sqrt{\mathrm{Re}(F(X,Y))^2 + \mathrm{Im}(F(X,Y))^2}$

fázis: $\phi(F(X,Y)) = \tan^{-1}\left(\frac{\mathrm{Im}(F(X,Y))}{\mathrm{Re}(F(X,Y))}\right)$

A frekvenciakép megjelenítése



Tulajdonságok

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

Ha $f(x,y)$ valós, akkor

$$F(X,Y) = \overline{F(-X,-Y)}$$

és

$$\left|F(X,Y)\right| = \left|F(-X,-Y)\right|$$

Tulajdonságok

linearitás:

$$f_1(x,y) \leftrightarrow F_1(X,Y)$$

$$f_2(x,y) \leftrightarrow F_2(X,Y)$$

$$a_1 \cdot f_1(x,y) + a_2 \cdot f_2(x,y) \leftrightarrow a_1 \cdot F_1(X,Y) + a_2 \cdot F_2(X,Y)$$

eltolás:

$$f(x,y) \leftrightarrow F(X,Y)$$

$$f(x-x_0, y-y_0) \leftrightarrow F(X,Y) \cdot e^{-2\pi i (x_0 X/M + y_0 Y/N)}$$

Tulajdonságok

skálázás:

$$f(x,y) \leftrightarrow F(X,Y)$$

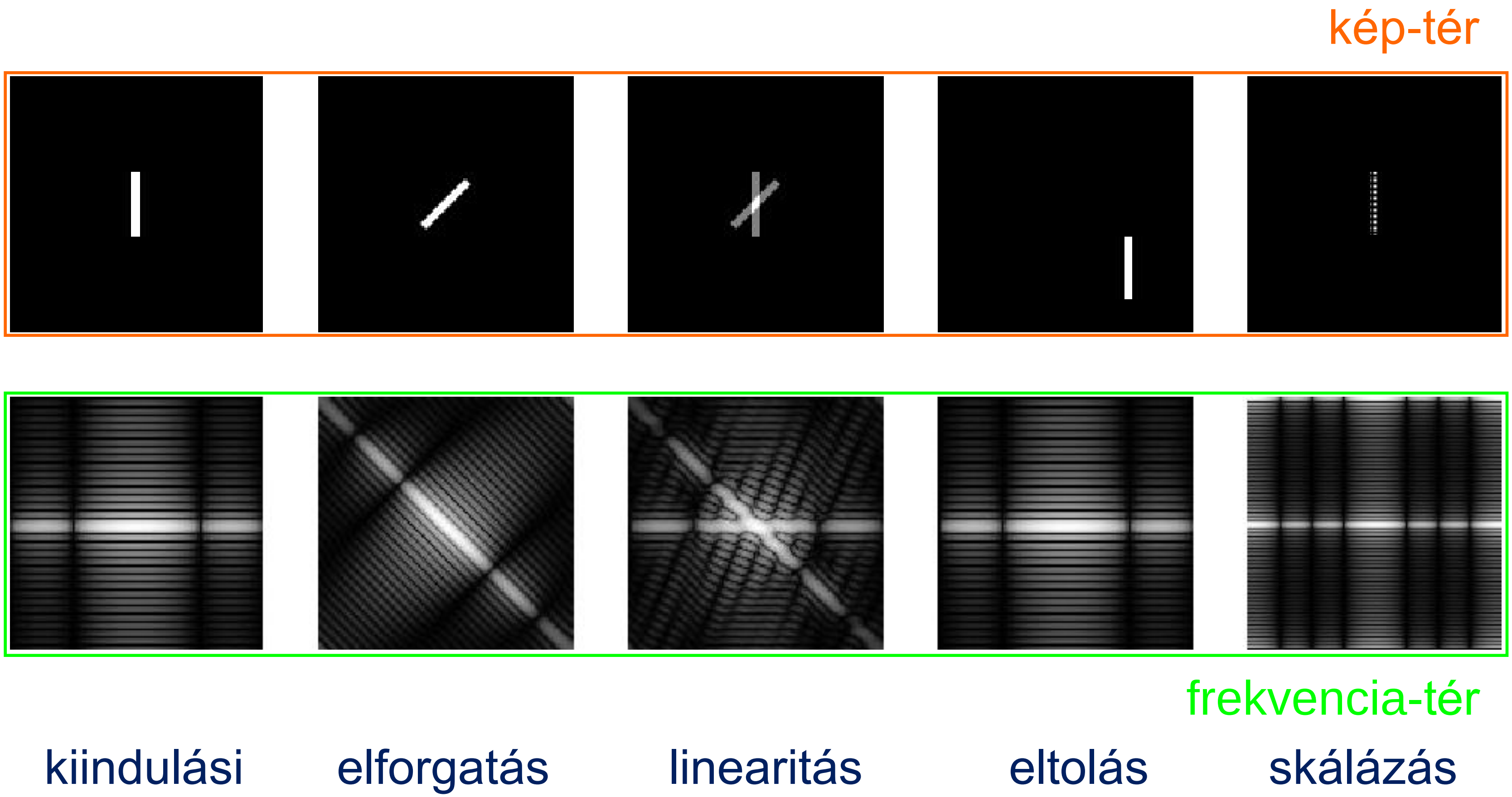
$$f(a\cdot x, b\cdot y) \leftrightarrow \frac{1}{|a\cdot b|}F\left(\frac{X}{a},\frac{Y}{b}\right)$$

elforgatás:

$$f(r,\varphi) \leftrightarrow F(R,\phi)$$

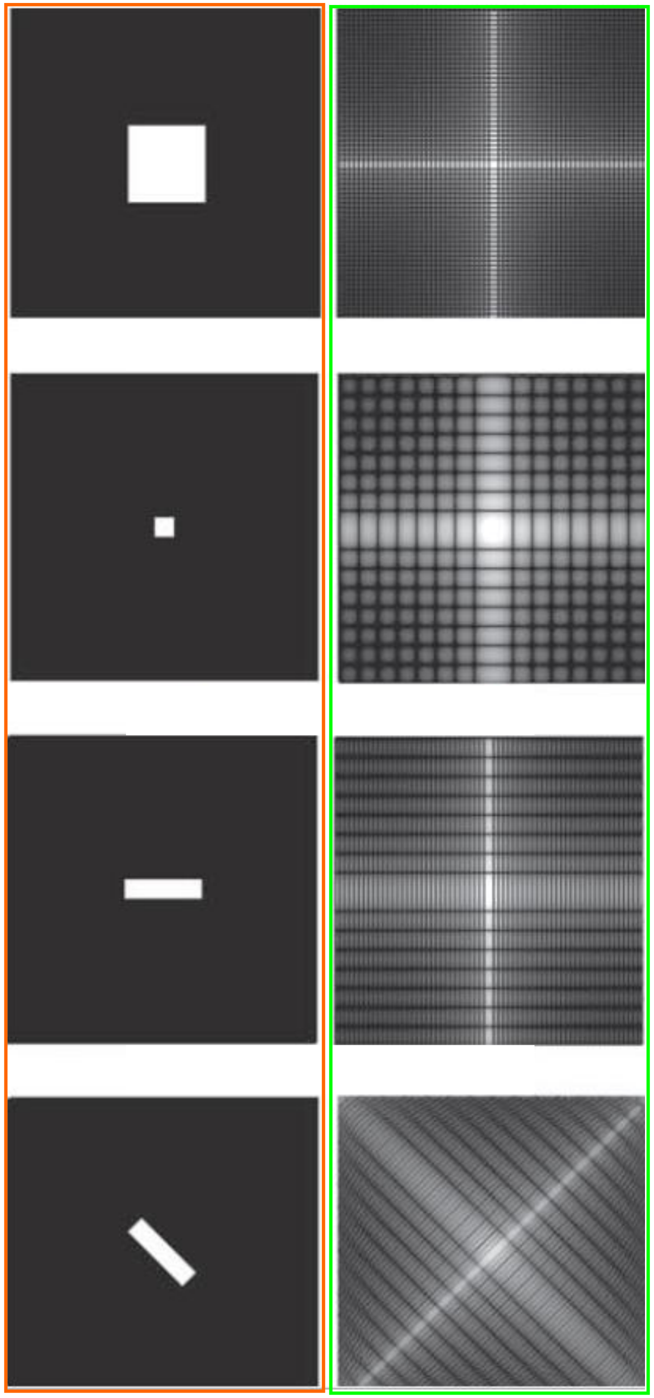
$$f(r,\varphi+\alpha) \leftrightarrow F(R,\phi+\alpha)$$

Tulajdonságok



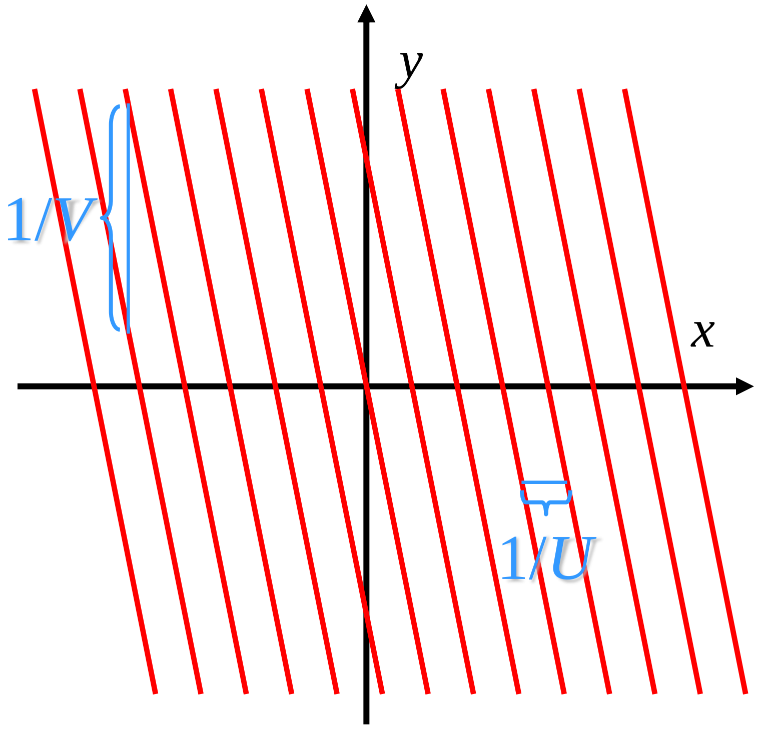
Tulajdonságok

kép-tér

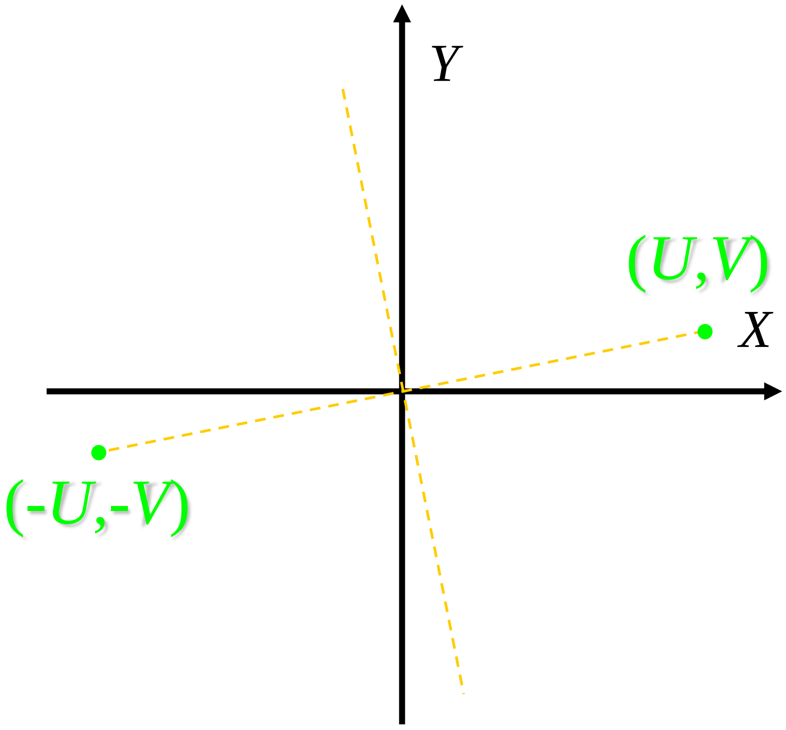


frekvencia-tér

Tulajdonságok



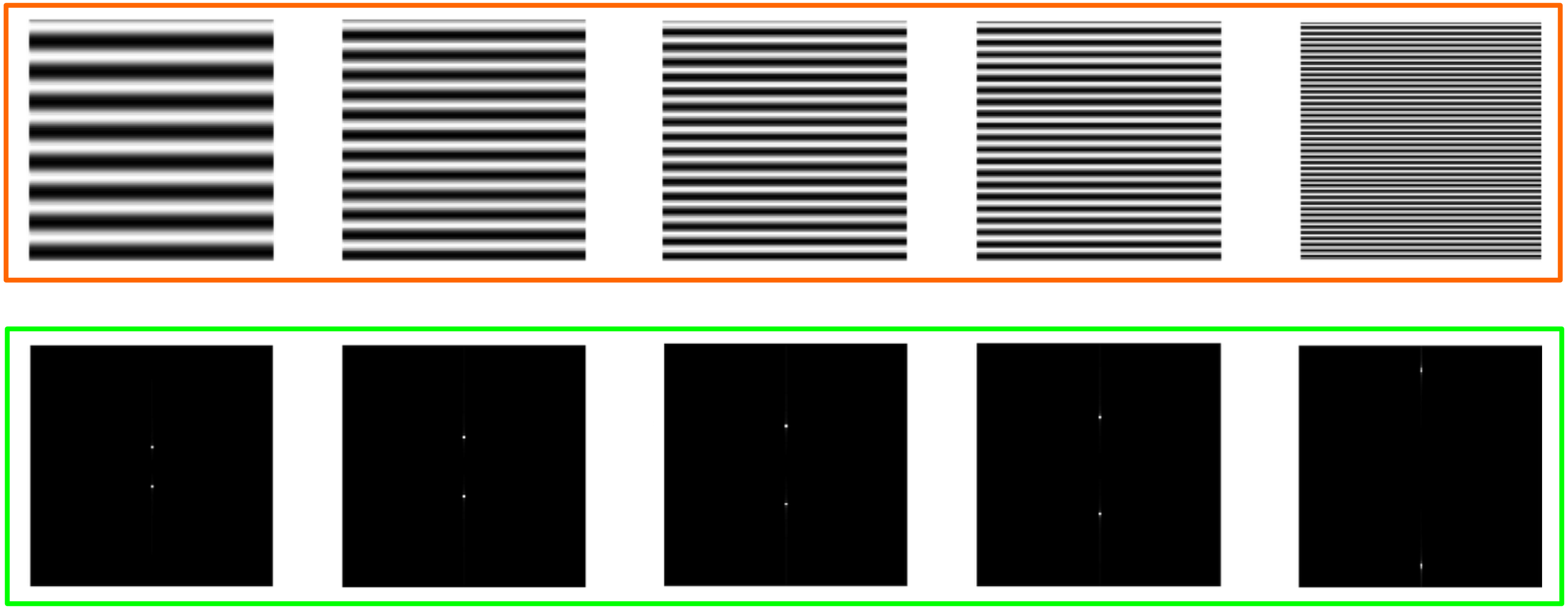
f



F

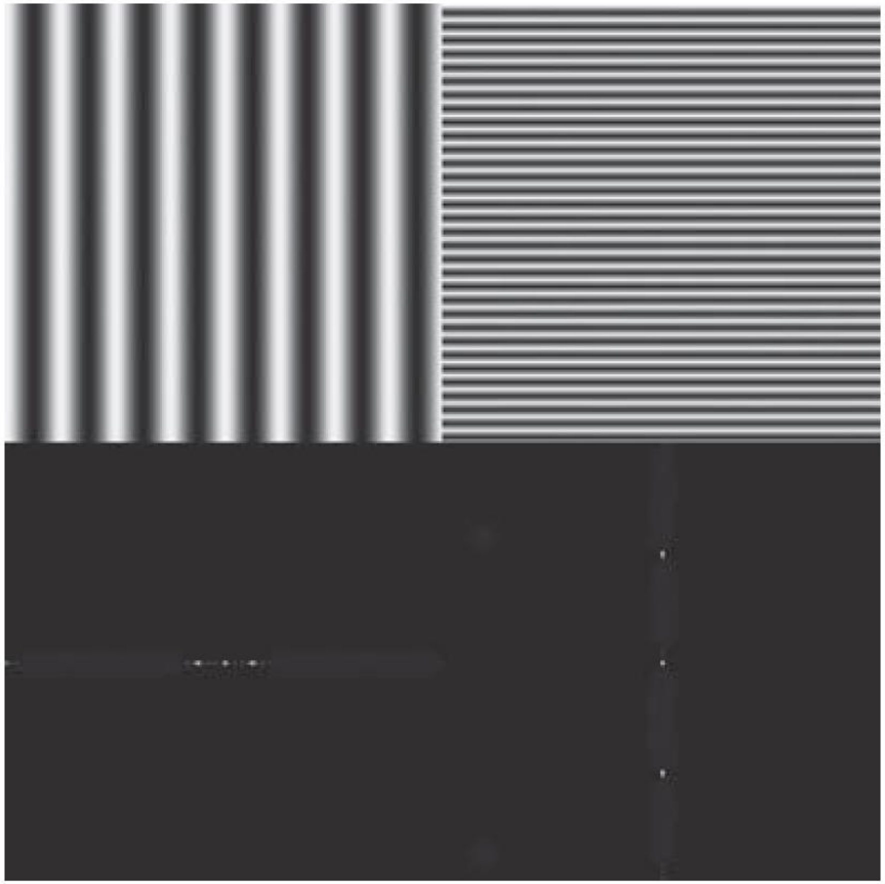
Tulajdonságok

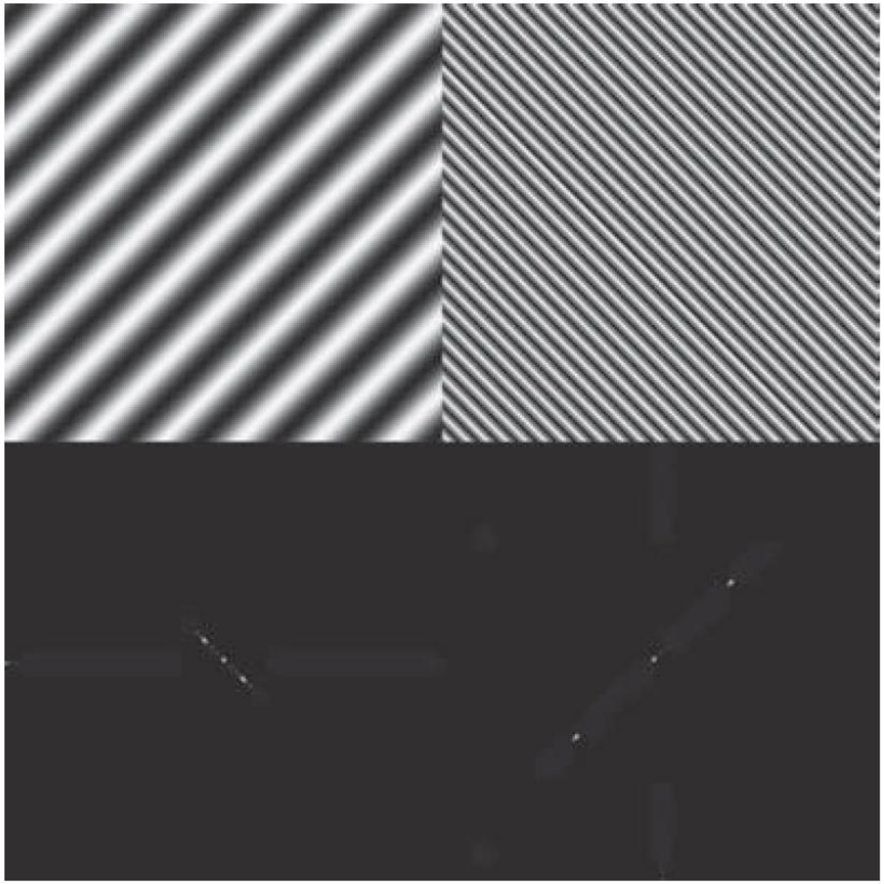
kép-tér



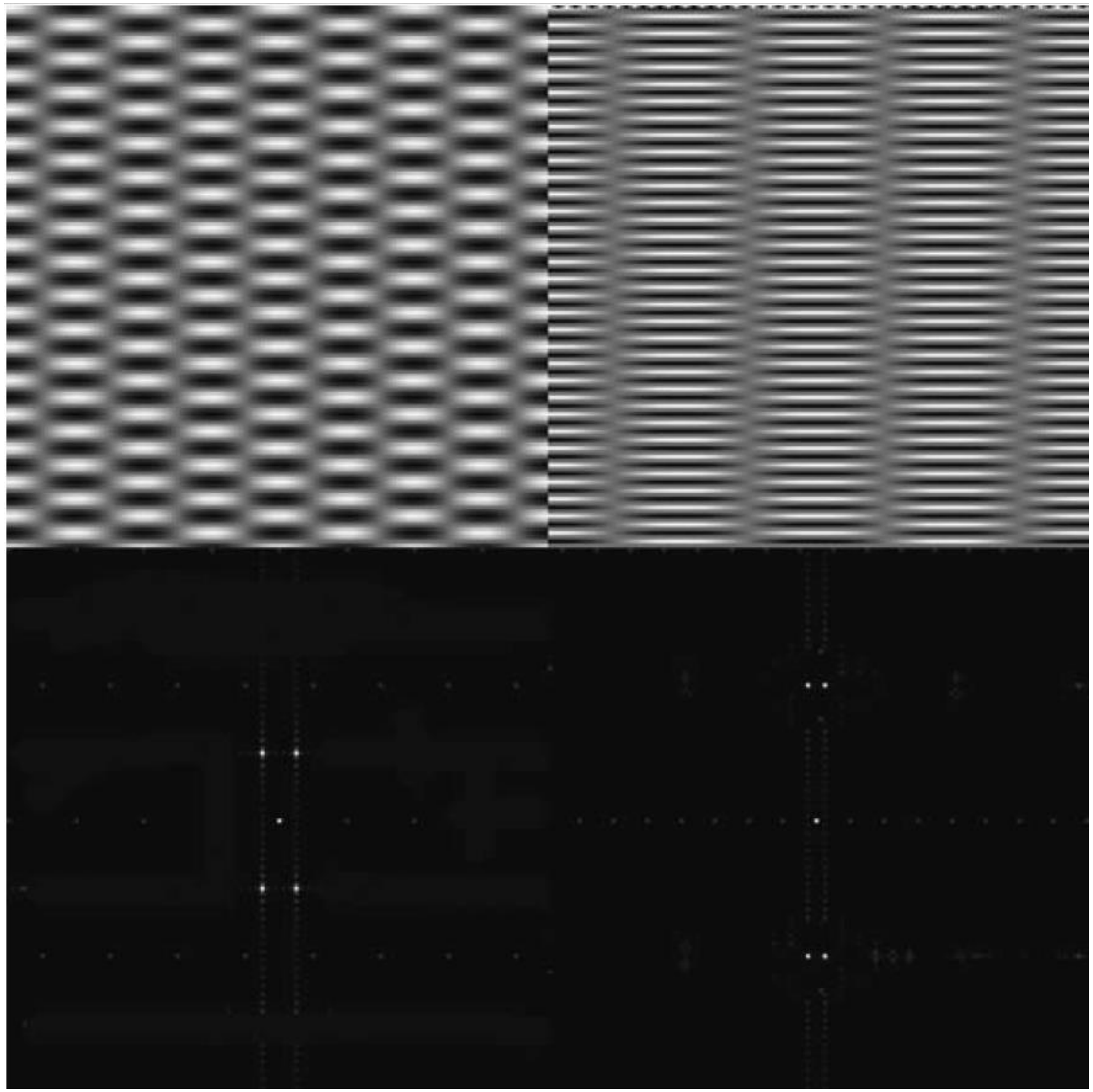
frekvencia-tér

Tulajdonságok



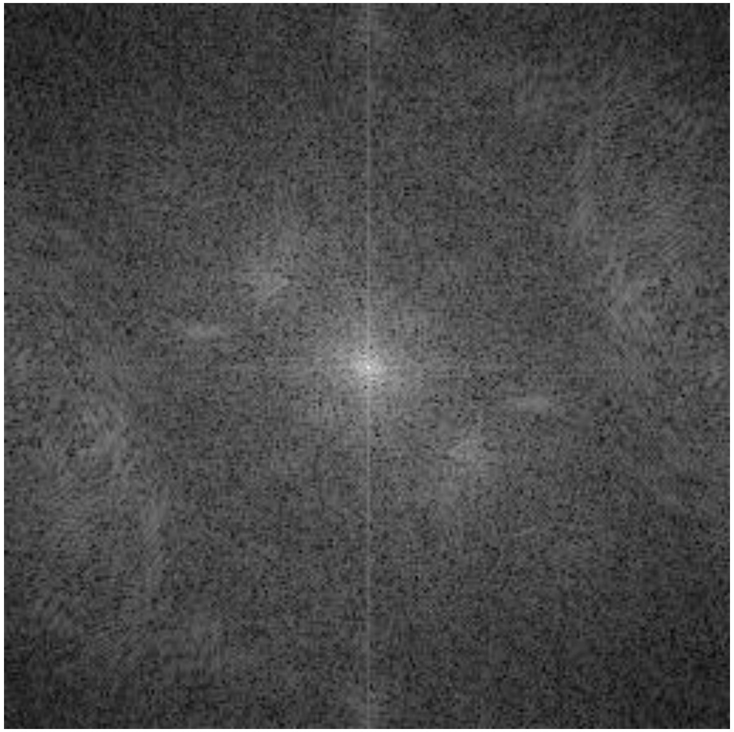


Tulajdonságok



Példa 2D Fourier transzformációra

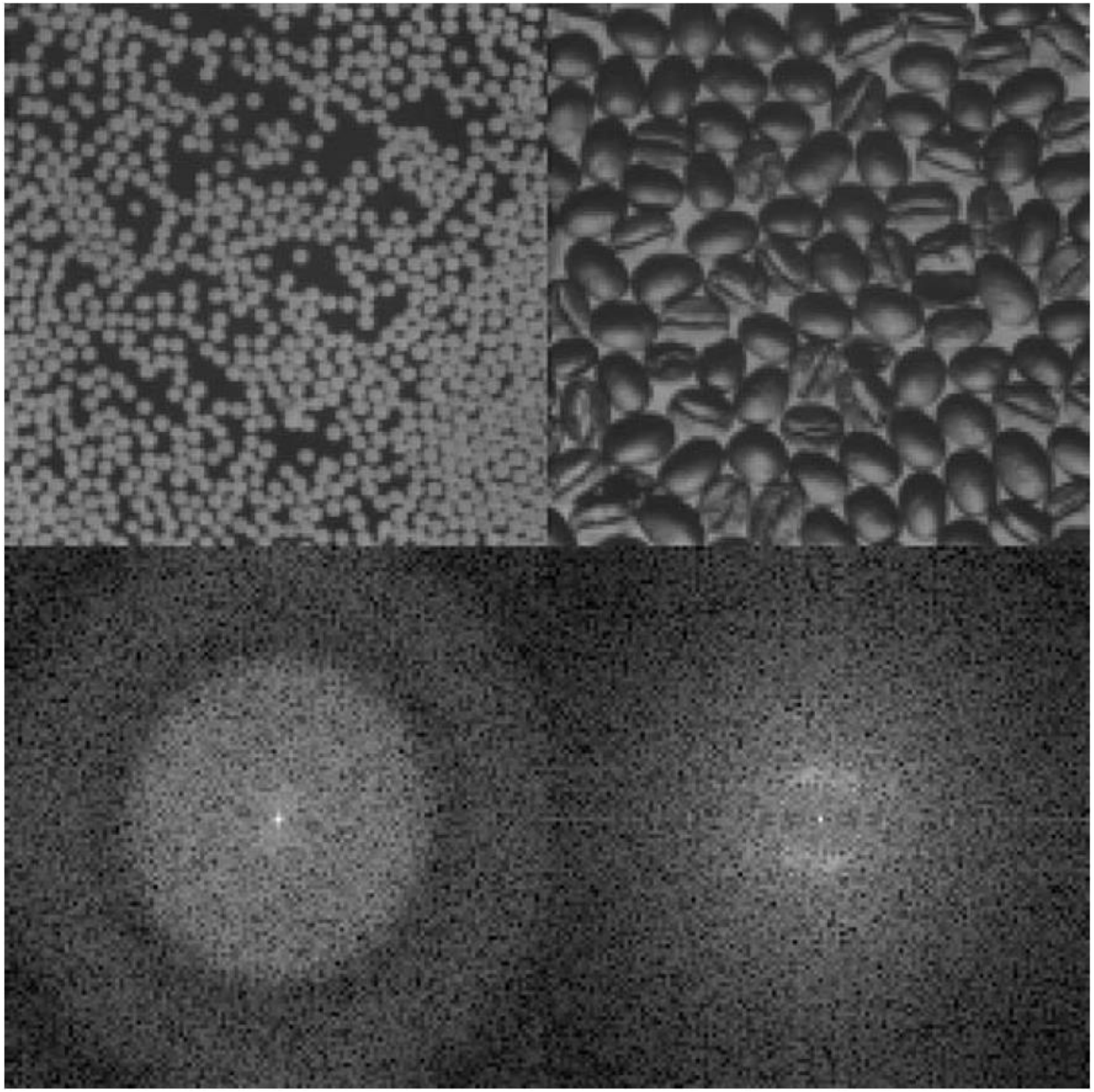
$\leftrightarrow$



$f(x)$ $|F(X,Y)|$

Példa 2D Fourier transzformációra

Példa 2D Fourier transzformációra



Analóg Fourier transzformáció - 1D

$$f \in L^1(-\infty, \infty)$$

(folytonos)

$$F(X) = \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i x X} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(X) \cdot e^{2\pi i x X} dX$$

(inverz trafó)

bázis-függvények

Analóg Fourier transzformáció - 2D

$$F(X,Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot e^{-2\pi i(xX+yY)}dxdy$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(X,Y)\cdot e^{2\pi i(xX+yY)}dXdY$$

bázis-függvények