Bernoulli-egyenlőtlenség

$a > -1$, $p > 0$, $p \neq 1$, $a \neq 0$.

$(1+a)^p > 1+ap$ $p>1$ esetén

$(1+a)^p < 1+ap$ $0<p<1$ esetén.

Ismert forma: $(1+a)^n > 1+na$ adott.

$1+2a+a^2 > 1+2a$ éppen látszik

n-re igaz

$$(1+a)^{n+1} > 1+na+a+na^2 = 1+(n+1)a+na^2 > 1+(n+1)a.$$

$(1+a)^p > 1+ap$ $p>1$.

$$1+a > \underbrace{(1+ap)^{1/p}\cdot 1^{1-\frac{1}{p}}}_{\text{súlyozott geom. közép}} \le \overbrace{\frac{1}{p}(1+ap)+1\left(1-\frac{1}{p}\right)}^{\text{súlyozott számtani közép}}$$

$$= \underbrace{\frac{1}{p}+a+1-\frac{1}{p}}_{1+a} \quad \text{ha } p>1.$$

Ha $0<p<1$:

$\cancel{(1+a)^p \cdot 1^{1-\frac{1}{p}}}$

$$(1+a)^p = (1+a)^p \cdot 1^{1-p} \le p(1+a)+(1-p)1 =$$

$$= p+pa+1-p = pa+1.$$

Csebisev-egyenlőtlenség:

$x_1, x_2, \dots, x_n$
$y_1, y_2, \dots, y_n$ azonosan rendezett számok

$x_i < x_j \Rightarrow y_i < y_j$.

$$\left.\begin{aligned}
x_1y_1+x_2y_2+\dots+x_ny_n &= x_1y_1+x_2y_2+\dots+x_ny_n \\
x_1y_1+x_2y_2+\dots+x_ny_n &\ge x_1y_2+x_2y_3+\dots+x_ny_1 \\
&\ge x_1y_3+x_2y_4+\dots+x_ny_2 \\
&\ \vdots \\
&\ge x_1y_n+x_2y_1+\dots+x_ny_{n-1}
\end{aligned}\right\} \ n \text{ db egyenlőtlenség}$$

Mindkét oldalt összeadva:

$$n(x_1y_1 + x_2y_2 + \dots + x_ny_n) \geq x_1(y_1 + \dots + y_n) + x_2(y_1 + \dots + y_n) + \dots + x_n(y_1 + \dots + y_n)$$

$$n(x_1y_1 + x_2y_2 + \dots + x_ny_n) \geq (x_1 + x_2 + \dots + x_n)(y_1 + \dots + y_n)$$

Ha azonosan rendezett.

Fordítottan rendezett esetben:

$$n(x_1y_1 + x_2y_2 + \dots + x_ny_n) \leq (x_1 + x_2 + \dots + x_n)(y_1 + y_2 + \dots + y_n).$$

$$S(x) = \frac{x_1 + \dots + x_n}{n} \qquad S(y) = \frac{y_1 + \dots + y_n}{n}$$

számtani közép

$$S(xy) = \frac{x_1y_1 + \dots + x_ny_n}{n}$$ lesz

$$\left.\begin{array}{l} S(xy) \geq S(x)\,S(y) \\ S(xy) \leq S(y)\,S(x) \end{array}\right\} \text{lesz}$$

Cauchy-féle egyenlőtlenség

$$|(x,y)| \leq \|x\|\,\|y\|.$$

$$x = (x_1, \dots, x_n)$$

$$y = (y_1, \dots, y_n)$$

$$|x_1y_1 + x_2y_2 + \dots + x_ny_n| \leq \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}\,\sqrt{y_1^2 + y_2^2 + \dots + y_n^2}.$$

$$(x + \lambda y, x + \lambda y) = (x_1 + \lambda y_1)^2 + (x_2 + \lambda y_2)^2 + \dots + (x_n + \lambda y_n)^2$$

$$\| \qquad f(\lambda) = \sum_{i=1}^{n} (x_i + \lambda y_i)^2.$$

$$f(\lambda) = \sum_{i=1}^{n} (x_i^2 + 2\lambda x_i y_i + \lambda^2 y_i^2) =$$

$$= \sum_{i=1}^{n} x_i^2 + 2\lambda \sum_{i=1}^{n} x_i y_i + \lambda^2 \sum_{i=1}^{n} y_i^2$$ ez egy parabola lesz

$A\lambda^2 + b\lambda + C \Rightarrow \sqrt{B^2 - 4ac}$ lesz a D.

$$4\left(\sum_{i=1}^{n} (x_i y_i)\right)^2 - 4\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \leq 0$$

ez épp a Csebisev-egyenlőtlenség alakja

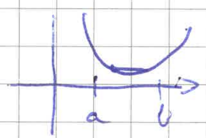
$x_1 + \lambda y_1 = 0 \qquad x = (x_1, \ldots, x_n)$

$\vdots \qquad\qquad\qquad x = -\lambda y$

$x_n + \lambda y_n = 0 \qquad y = (y_1, \ldots, y_n)$

Jensen-egyenlőtlenség:

$f(x)$ konvex $[a, b]$-n, $a < b$



húr alatt

$f(x)$ konkáv, ha a húr felett marad

$f(x)$ konvex $[a, b]$-n, ha $\forall x_1, x_2 \in [a, b]$: $f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$.

$\forall x_1, x_2, \ldots, x_n \in [a, b]$, $x_i \neq x_j$

$p_1, p_2, \ldots, p_n > 0$, $\sum_i^n p_i = 1$, $f(p_1 x_1 + p_2 x_2 + \ldots + p_n x_n) \leq p_1 f(x_1) + p_2 f(x_2) + \vdots\; p_n f(x_n)$.

$x_1 t + x_2 (1-t)$, $0 \leq t \leq 1$.

$$f(x_1 t + x_2(1-t)) \leq t f(x_1) + (1-t) f(x_2)$$

Konkáv esetben fordítva van

Bizonyítása:

Tfh 2-re igaz.

$$\frac{x_1+x_2+x_3+x_4}{4} = \frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}}{2}$$

$$f\left(\frac{x_1+x_2+x_3+x_4}{4}\right) \le \frac{f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)}{2} \le \frac{\frac{f(x_1)+f(x_2)}{2}+\frac{f(x_3)+f(x_4)}{2}}{2}$$

innen:

2^n tagra igaz. 2^{n+1}

$$\frac{x_1+x_2+\dots+x_{2^n}+x_{2^n+1}+\dots+x_{2^{n+1}}}{2^{n+1}} = \frac{\frac{x_1+\dots+x_{2^n}}{2^n}+\frac{x_{2^n+1}+\dots+x_{2^{n+1}}}{2^n}}{2^n}$$

$$f\left(\frac{x_1+\dots+x_k}{k}\right) = f\left(\frac{x_1+\dots+x_k+A+\dots+A}{2^n}\right) = f\left(\frac{k\overset{A}{\cancel{x}}+(2^n-(k+1)^{+1})A}{2^n}\right) = f(A)$$

$$A = \frac{x_1+\dots+x_k}{k}$$

$$\le \frac{f(x_1)+\dots+f(x_k)+f(A)(2^n-k)}{2^n}$$

$$2^n f(A) - f(A)(2^n-k) \le \frac{f(x_1)+\dots+f(x_k)}{\cancel{2^n}}$$

$$f(A)k \le f(x_1)+\dots+f(x_k)$$

$$f(A) \le \frac{f(x_1)+\dots+f(x_k)}{k}$$

$$\frac{p_1x_1+\dots+p_nx_n}{p_1+\dots+p_n} \quad p_i \text{ egész.}$$

$$p_i=\frac{r_i}{q_i} \quad \text{racionális} \qquad p_i=\frac{s_i}{q} \quad \text{alakúak}$$

$$\sum_{i=1}^{n}\frac{p_ix_i}{p_1+\dots+p_n}=\sum\frac{s_ix_i}{s_1+\dots+s_n}.$$

$$\sum\frac{r_ix_i}{r_1+\dots+r_n}$$ a függvény folytonos. Nevezetes egyenlőtlenségek.

Példa

A $\sin(x)$ $[0,\pi]$-n konkáv. Lássuk be.

$\forall x_1,x_2\in[0,\pi]$ $\cancel{\sin(x_1)+}$ $\sin\left(\frac{x_1+x_2}{2}\right) \geq \frac{\sin x_1+\sin x_2}{2}$ kell.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\sin(\underbrace{\alpha+\beta}_{x_1})+\sin(\underbrace{\alpha-\beta}_{x_2})=2\sin\alpha\cos\beta$$

$$\sin x_1+\sin x_2=2\sin\frac{x_1+x_2}{2}\cos\frac{x_1-x_2}{2}$$

$$0 \geq \frac{\sin x_1+\sin x_2}{2}-\sin\frac{x_1+x_2}{2}=$$

$$=\sin\frac{x_1+x_2}{2}\left(\cos\frac{x_1-x_2}{2}-1\right) \quad \square$$

Példa α,β,γ egy tetszőleges háromszög szögei.

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\leq\frac{1}{8}.$$

$\gamma = 180^\circ - (\alpha + \beta)$

$$\sqrt[3]{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} < \frac{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}}{3} < \underbrace{\frac{3\sin\left(\frac{\alpha+\beta+\gamma}{3\cdot 2}\right)}{3}}_{\frac{1}{2}}$$

konkáv

$$\frac{\sin x_1 + \sin x_2 + \sin(x_3)}{3} < \sin\left(\frac{x_1+x_2+x_3}{3}\right) \quad \square$$

Következő példa:

$$\cos\alpha + \cos\beta + \cos\gamma \le \frac{3}{2} \qquad \Delta \text{ van.}$$

① $0 < \alpha, \beta, \gamma \le \frac{\pi}{2}$. $\qquad \alpha+\beta+\gamma = \pi$.

$$\frac{\cos\alpha + \cos\beta + \cos\gamma}{3} \le \underbrace{\cos\left(\frac{\alpha+\beta+\gamma}{3}\right)}_{\cos(60^\circ)} = \frac{1}{2} \quad \square$$

② $\frac{\pi}{2} < \gamma$ de $\alpha, \beta < \frac{\pi}{2}$

$\alpha + \beta < \frac{\pi}{2}$

$$\cos\alpha + \cos\beta + \cos\gamma \le 2\cos\frac{\alpha+\beta}{2} + \cos(\pi - (\alpha+\beta)) =$$

$$= 2\cos\frac{\alpha+\beta}{2} + \cos(\alpha+\beta)$$

$$= 2\cos\frac{\alpha+\beta}{2} - 2\cos^2\frac{\alpha+\beta}{2} - 1.$$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$1 = \cos^2\alpha + \sin^2\alpha$ $\qquad = -2(\cos$

$\cos 2\alpha + 1 = 2\cos^2\alpha$

$\vdots$

$1 + \cos(\alpha+\beta) = 2\cos^2\left(\frac{\alpha+\beta}{2}\right)$ $\qquad$ elbasztuk

$$a^a b^b c^c \geqslant \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

$$a^{\frac{a}{a+b+c}} b^{\frac{b}{a+b+c}} c^{\frac{c}{a+b+c}}$$

$$\frac{1}{a^{\frac{a}{a+b+c}} b^{\frac{b}{a+b+c}} c^{\frac{c}{a+b+c}}} = \left(\frac{1}{a}\right)^{\frac{a}{a+b+c}} \left(\frac{1}{b}\right)^{\frac{b}{a+b+c}} \left(\frac{1}{c}\right)^{\frac{c}{a+b+c}} \leq$$

$$\frac{1}{a}\frac{a}{a+b+c} + \frac{1}{b}\frac{b}{a+b+c} + \frac{1}{c}\frac{c}{a+b+c} = \frac{3}{a+b+c}.$$

pl.

$$\boxed{n^n > (n+1)^{n-1} \quad \text{ha} \quad n>1, \; n \in \mathbb{R}.}$$

$$n^n(n+1) > (n+1)^n$$

$$n^{n+1} + n^n > (n+1)^n. \quad \text{nem jó}$$

$$(1+a)^p > 1+ap \qquad a>-1. \qquad p>1. \qquad \text{Bernoulli}$$

$$n^n = (1+n-1)^n >$$

$$a = n-1$$

$$n^{\frac{n}{n-1}} = (1+n-1)^{\frac{n}{n-1}} > 1+(n-1)\frac{n}{n-1} = 1+n$$

$$a = n-1 \qquad p = \frac{n}{n-1}$$

$$\Downarrow$$

$$n^n > (1+n)^{\cancel{n+1}\, n-1}. \quad \square$$

$$2^{-\cos^2 x} + 2^{-\sin^2 x} = \sin y + \cos y$$

$$\frac{2^{-\cos^2 x} + 2^{-\sin^2 x}}{2} \geq \sqrt{2^{-\cos^2 x} \cdot 2^{-\sin^2 x}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 2^{-\cos^2 x} + 2^{-\sin^2 x} \geq \sqrt{2}$$

$$\frac{\sin y + \cos y}{2} \leq \sqrt{\frac{\sin^2 y + \cos^2 y}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\left.\begin{array}{l} \sin^2 x = \cos^2 x \\ \sin y = \cos y \end{array}\right\}$$

2 egyenlet, 3 ismeretlennel.

$x + y + z = 11$ számtani közép

$x^2 + 2y^2 + 3z^2 = 66$ súlyozott négyzetes közép

$$\frac{36}{66}\left(\frac{x}{6}\right)^2 + \frac{18}{66}\left(\frac{y}{3}\right)^2 + \frac{12}{66}\left(\frac{z}{2}\right)^2 = 1 \geq \left(\frac{36}{66}\frac{x}{6} + \frac{18}{66}\frac{x}{3} + \frac{12}{66}\frac{z}{2}\right)^2$$

$$\left(\frac{x}{11} + \frac{y}{11} + \frac{z}{11}\right)^2 = 1$$

$$\frac{x}{6} = \frac{y}{3} = \frac{z}{2} \qquad x = 3z \qquad y = \frac{3}{2}z$$

$$3z + \frac{3}{2}z + z = 11$$

—o—

Oldjuk meg

$$(\sin^4 x + 1)(\cos^4 x + 1) = \frac{25}{16}\sin^4 2x.$$

$$(\sin^4 x + 1)(\cos^4 x + 1) = \frac{25}{16}\sin^4 2x > \frac{25}{16}\, 2^4 \sin^4 x \cos^4 x$$

$$\left(1+\frac{1}{\sin^4 x}\right)\left(1+\frac{1}{\cos^4 x}\right) = 25$$

$$1 + \frac{1}{\sin^4 x} + \frac{1}{\cos^4 x} + \frac{1}{\sin^4 x \cos^4 x} = 25$$

$$\frac{1}{\sin^4 x} + \frac{1}{\cos^4 x} \geq 2\sqrt{\frac{1}{\sin^4 x}\,\frac{1}{\cos^4 x}}$$

$$1 + \frac{1}{\sin^4 x} + \frac{1}{\cos^4 x} + \frac{1}{\sin^4 x \cos^4 x} \geq 1 + \frac{2}{\sin^2 x \cos^2 x} + \left(\frac{1}{\sin^2 x \cos^2 x}\right)^2 = \dots$$

— o —

<u>Példa</u> $p > 0,\ q > 0,\ x > 0,\ y > 0.$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$$

$$xy = \left(x^p\right)^{\frac{1}{p}}\left(y^q\right)^{\frac{1}{q}} \leq \frac{1}{p}x^p + \frac{1}{q}y^q. \ \square$$

— o —

$a_1, a_2, \dots, a_n > 1.$

$$\left(a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n}\right)^n \geq (a_1 a_2 \cdots a_n)^{\cancel{n+1}\ a_1+a_2+\dots+a_n}$$

$$n\cdot\left(a_1 \ln a_1 + a_2 \ln a_2 + \dots + a_n \ln a_n\right) \geq (a_1+a_2+\dots+a_n)(\ln a_1 + \ln a_2 + \dots + \ln a_n)$$

Csebisev

— o —

Ha $n > 1 \Rightarrow n! < \left(\frac{n+1}{2}\right)^n.$

$$\sqrt[n]{1\cdot 2\cdot \ldots \cdot n} \le \frac{1+2+\ldots+n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$$

— o —

$a, b, c > 0$. Legyen.

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} \le \frac{1}{a^3}\sqrt{\frac{b}{a}}+\frac{1}{b^3}\sqrt{\frac{c}{b}}+\frac{1}{c^3}\sqrt{\frac{a}{c}}.$$

$$\underbrace{\frac{1\sqrt{a}}{a^3\sqrt{a}}+\frac{1\sqrt{b}}{b^3\sqrt{b}}+\frac{1\sqrt{c}}{c^3\sqrt{c}}}_{\text{skalárszorzat}}$$

$\sqrt{b}$, $\sqrt{c}$, $\sqrt{a}$

$\frac{1}{a^3\sqrt{a}}$, $\frac{1}{b^3\sqrt{b}}$, $\frac{1}{c^3\sqrt{c}}$) két vekt

$\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$

$\cancel{a^3\sqrt{a}, b^3\sqrt{b}, c^3\sqrt{c}}$

$\frac{1}{a^3\sqrt{a}}$, $\frac{1}{b^3\sqrt{b}}$, $\frac{1}{c^3\sqrt{c}}$) két vekt

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$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \ge \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

Cauchy-t lehetne még.

$$\frac{x}{y}1+\frac{y}{z}\cdot 1+\frac{z}{x}\cdot 1 \le \sqrt{\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}}\,\sqrt{3}$$

$$\frac{\frac{x}{y}+\frac{z}{x}+\frac{y}{z}}{3} \ge \sqrt[3]{1} = 1$$

$$\frac{\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}}{3} \ge \left(\frac{\left|\frac{x}{y}\right|+\left|\frac{y}{z}\right|+\left|\frac{z}{x}\right|}{3}\right)^2$$

$$\frac{\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}}{3} \ge \left(\frac{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}{3}\right)^2 \ge \ldots$$