(1/A) Igazoljuk def. szerint: $\lim\limits_{n\to\infty} \frac{3n-2}{2n+5} = \frac{3}{2}$

(D:) Legyen $\varepsilon > 0$ tetszőleges. Kell: $\nu = \nu(\varepsilon)$, hogy

$n > \nu(\varepsilon)$ esetén

$$\left|\frac{3n-2}{2n+5} - \frac{3}{2}\right| < \varepsilon \qquad \left|\frac{(3n-2)\cdot 2 - 3\cdot(2n+5)}{(2n+5)\cdot 2}\right| = \left|\frac{\cancel{6n} - 4 - \cancel{6n} - 15}{4n+10}\right| =$$

$$= \left|\frac{-19}{4n+10}\right| = \frac{19}{4n+10} < \varepsilon \iff \frac{19}{\varepsilon} < 4n+10 \iff \frac{19}{\varepsilon} - 10 < 4n \iff$$

$$\iff \underbrace{\frac{1}{4}\cdot\left(\frac{19}{\varepsilon} - 10\right)} < n.$$

$$\frac{19}{4\varepsilon} - \frac{10}{4} < n \implies \nu(\varepsilon) = \underline{\underline{\frac{19}{4\varepsilon} - \frac{10}{4}}}$$

(1/B) Igazoljuk def. szerint: $\lim\limits_{n\to\infty} \frac{\sqrt{n^4 - 3n + 4}}{3n+2} = \infty.$

(D:) Legyen $K \in \mathbb{R}$ tetszőleges. Kell: $\nu = \nu(K)$, hogy

$n > \nu(K)$ esetén

$$\underbrace{\frac{\sqrt{n^4 - 3n + 4}}{3n+2}}_{\text{alulról becsüljük.}} > K \qquad 3n+2 \le 3n+2n = 5\underline{\underline{n}}.$$

$$n^4 - 3n + 4 \geq n^4 - 3n \geq n^4 - \frac{1}{3}n^4 = \frac{2}{3}n^4.$$

$$\Uparrow$$

$$3n < \frac{1}{3}n^4$$
$$9n < n^4$$
$$9 < n^3$$
$$\sqrt[3]{9} < n$$

így $\frac{\sqrt{n^4 - 3n + 4}}{3n+2} > \frac{\sqrt{\frac{2}{3}n^4}}{5n} = \frac{\sqrt{\frac{2}{3}}\, n^2}{5\cancel{n}} =$

$$= \sqrt{\frac{2}{3}}\cdot\frac{1}{5}\, n > K.$$

$$n > \frac{5K}{\sqrt{\frac{2}{3}}}$$

így: $\nu(K) = \max\left\{\sqrt[3]{9};\ \frac{5K}{\sqrt{\frac{2}{3}}}\right\}.$

(1/C) Igazoljuk def. szerint: $\lim\limits_{n\to\infty} \frac{n^2 - 5n}{-n+3} = -\infty.$

(D:) Legyen $k \in \mathbb{R}$. Kell $\nu = \nu(k)$, hogy $n > \nu(k)$ esetén

$$\frac{n^2 - 5n}{-n+3} < k.$$

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$$\frac{n^2-5n}{-n+3} = -\underbrace{\frac{n^2-5n}{n-3}}_{\text{pozitív tehát}} < \varepsilon$$

$$\frac{n^2-5n}{n-3} > -\varepsilon.$$

⇑ alulról becslés kell!

$$n^2-5n > n^2-\frac{1}{2}n^2 = \frac{1}{2}n^2$$

⇑

$$5n < \frac{1}{2}n^2$$

$$10n < n^2$$

$$\boxed{10 < n}$$

így: $\frac{n^2-5n}{n-3} > \frac{\frac{1}{2}n^2}{n-3} > \frac{\frac{1}{2}n^2}{n} = \frac{1}{2}n > -\varepsilon$

$n-3 < n$ $\qquad n > -2\varepsilon.$

így: $\nu(\varepsilon) = \max\{10; -2\varepsilon\}.$

2/A

$$\lim_{n\to\infty}\frac{-n^4-3n}{3n^2+11} = \lim_{n\to\infty}\frac{n^4}{n^2}\cdot\frac{-1-\frac{3}{n^3}}{3+\frac{11}{n^2}} = \lim_{n\to\infty} n^2\cdot\frac{-1-\frac{3}{n^2}}{3+\frac{11}{n^2}} = \infty^2\cdot\frac{-1-0}{3+0} =$$

$$= \underline{\underline{-\infty}}$$

2/B

$$\lim_{n\to\infty}\frac{n^3+2n-1}{n^3-n^2} = \lim_{n\to\infty}\frac{n^3}{n^3}\cdot\frac{1+\frac{2}{n^2}-\frac{1}{n^3}}{1-\frac{1}{n}} = 1\cdot\frac{1+0-0}{1-0} = \underline{\underline{1}}$$

2/C

$$\lim_{n\to\infty}\frac{\sqrt{n^2-3n+1}}{\sqrt{n-5}} = \lim_{n\to\infty}\frac{\sqrt{n^2}}{\sqrt{n}}\cdot\frac{\sqrt{1-\frac{3}{n}+\frac{1}{n^2}}}{\sqrt{1-\frac{5}{n}}} = \lim_{n\to\infty}\underbrace{\frac{n}{\sqrt{n}}}_{\sqrt{n}}\cdot\frac{\sqrt{1-\frac{3}{n}+\frac{1}{n^2}}}{\sqrt{1-\frac{5}{n}}} =$$

$$= \infty\cdot\frac{\sqrt{1-0+0}}{\sqrt{1-0}} = \underline{\underline{\infty}}$$

2/D

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n-1}\right) = \lim_{n\to\infty}\sqrt{n}\,\frac{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}{\sqrt{n+1}+\sqrt{n-1}} = \lim_{n\to\infty}\frac{\sqrt{n}\cdot((n+1)-(n-1))}{\sqrt{n+1}+\sqrt{n-1}} =$$

$$= \lim_{n\to\infty}\frac{\sqrt{n}\cdot(2)}{\sqrt{n+1}+\sqrt{n-1}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}}\cdot\frac{2}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}} = 1\cdot\frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \underline{\underline{1}}$$

(2/E)

$$\lim_{n\to\infty}\frac{(-1)^n\cdot 2^n+5\cdot 6^n}{3^{n+1}+6^{n-1}}=\lim_{n\to\infty}\frac{\frac{1}{6^n}}{\frac{1}{6^n}}\cdot\frac{(-1)^n\cdot 2^n+5\cdot 6^n}{3^{n+1}+6^{n-1}}=\lim_{n\to\infty}\frac{5+(-1)^n\cdot\left(\frac{2}{6}\right)^n}{(3^n\cdot 3^1+6^n\cdot 6^{-1})\cdot\frac{1}{6^n}}=$$

$$=\lim_{n\to\infty}\frac{5+(-1)^n\cdot\left(\frac{2}{6}\right)^n}{\left(\frac{3}{6}\right)^n\cdot 3+\left(\frac{6}{6}\right)^n\cdot\frac{1}{6}}=\frac{5+0}{0+\frac{1}{6}}=6\cdot 5=\underline{\underline{30}}$$

(2/F)

$$\lim_{n\to\infty}\sqrt[n]{7^n-2\cdot 5^n+1}=?$$

Ha n elég nagy: $\sqrt[n]{7^n-2\cdot 5^n+1}\approx\sqrt[n]{7^n}=7.$

Rendőrelvvel kell ezt belátni.

$$7^n-2\cdot 5^n+1>7^n-2\cdot 5^n>7^n-\frac{1}{2}7^n=\frac{1}{2}7^n.$$

$$2\cdot 5^n<\frac{1}{2}7^n$$

$$4\cdot 5^n<7^n$$

$$7^n-2\cdot 5^n+1<7^n+1<7^n+7^n=\underline{2\cdot 7^n}$$

$$2\cdot 7^n\geq 7^n-2\cdot 5^n+1\geq\frac{1}{2}7^n$$

$$\Updownarrow$$

$$\sqrt[n]{2\cdot 7^n}\geq\sqrt[n]{7^n-2\cdot 5^n+1}\geq\underbrace{\sqrt[n]{\frac{1}{2}7^n}}$$

$$\underbrace{\sqrt[n]{2}}_{1}\cdot\underbrace{\sqrt[n]{7^n}}_{7}\ \text{Ha } n\to\infty \qquad \underbrace{\sqrt[n]{\frac{1}{2}}}_{1}\cdot\underbrace{\sqrt[n]{7^n}}_{7}\ \text{Ha } n\to\infty$$

Ha $n\to\infty$

$$\underline{\underline{7}}$$

(2/G)

$$\lim_{n\to\infty}\sqrt[n]{n^5-3n^3+2}=?$$

Ha n elég nagy: $\sqrt[n]{n^5-3n^3+2}\approx\sqrt[n]{n^5}=1$ Ezt sejtjük.

$$n^5-3n^3+2 > n^5-3n^3 > n^5-\frac{1}{3}n^5 = \underline{\frac{2}{3}n^5}$$

$$3n^3 < \frac{1}{3}n^5$$

$$9n^3 < n^5$$

$$9 < n^2$$

$$3 < n$$

$$n^5-3n^3+2 < n^5+2 < n^5+2n^5 = \underline{3n^5}$$

$$3n^5 > n^5-3n^3+2 > \frac{2}{3}n^5$$

$$\underbrace{\sqrt[n]{3n^5}}_{\sqrt[n]{3}\cdot\left(\sqrt[n]{n}\right)^5} > \sqrt[n]{n^5-3n^3+2} > \underbrace{\sqrt[n]{\frac{2}{3}n^5}}_{\sqrt[n]{\frac{2}{3}}\cdot\left(\sqrt[n]{n}\right)^5}$$

Ha $n\to\infty$: $\sqrt[n]{3}\to 1$, $\sqrt[n]{n}\to 1$; $\sqrt[n]{\frac{2}{3}}\to 1$, $\sqrt[n]{n}\to 1$

Ha $n\to\infty$: $\sqrt[n]{n^5-3n^3+2}\to \underline{1}.$

(2/K)

$$\lim_{n\to\infty}\left(\frac{4n-3}{5n+11}\right)^{n-2} = ?$$

$$\lim_{n\to\infty}\frac{4n-3}{5n+11} = \frac{4}{5} \quad \text{és} \quad \lim_{n\to\infty}(n-2) = \infty.$$

$$\frac{4}{5} = \frac{8}{10}$$

$$\frac{7}{10} < \frac{4n-3}{5n+11} < \frac{9}{10}$$

$$\left(\frac{7}{10}\right)^{n-2} < \left(\frac{4n-3}{5n+11}\right)^{n-2} < \left(\frac{9}{10}\right)^{n-2}$$

$$\underbrace{\underbrace{\left(\frac{7}{10}\right)^{n}}_{\to 0}\cdot\left(\frac{7}{10}\right)^{-2}}_{\to 0} < \left(\frac{4n-3}{5n+11}\right)^{n-2} < \underbrace{\underbrace{\left(\frac{9}{10}\right)^{n}}_{\to 0}\cdot\left(\frac{9}{10}\right)^{-2}}_{\to 0}$$

Ha $n\to\infty$: $\left(\frac{4n-3}{5n+11}\right)^{n-2}\to 0.$

2/M)

$$\lim_{n\to\infty}\left(\frac{4n+2}{3n-9}\right)^{3n}=?$$

$$\lim_{n\to\infty}\frac{4n+2}{3n-9}=\frac{4}{3}>1.$$

$$\frac{4}{3}=\frac{8}{6}\quad\Rightarrow\quad\frac{4n+2}{3n-9}>\frac{7}{6}$$

$$\Downarrow$$

$$\underbrace{\left(\frac{4n+2}{3n-9}\right)^{3n}}_{\downarrow\ \infty.}>\left(\frac{7}{6}\right)^{3n}=\underbrace{\left[\left(\frac{7}{6}\right)^{3}\right]^{n}}_{\downarrow\ \infty}$$

2/L)

$$\lim_{n\to\infty}\left(\underbrace{\frac{n+1}{n-2}}_{\to 1,\ \text{ha } n\to\infty.}\right)^{3n-5}=?$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n-2}\right)^{3n-5}=\lim_{n\to\infty}\left(\frac{n-2+3}{n-2}\right)^{3n-5}=\lim_{n\to\infty}\left(1+\frac{3}{n-2}\right)^{3n-5}=$$

$$=\lim_{n\to\infty}\left(1+\frac{3}{n-2}\right)^{3n-5}=\lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n-2}{3}}\right)^{\frac{n-2}{3}}\right]^{\frac{3}{n-2}\cdot 3n-5}=$$

$$=\lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{\frac{n-2}{3}}\right)^{\frac{n-2}{3}}}_{\to e\ \text{Ha } n\to\infty}\right]^{\underbrace{\frac{9n-15}{n-2}}_{9\ \text{Ha } n\to\infty}}=e^{9}.$$

2/N)

$$\lim_{n\to\infty}\sqrt[n]{\frac{2n^2-3n+1}{3n^2+11}}=?$$

Ha $n\to\infty$: $\quad\dfrac{2n^2-3n+1}{3n^2+11}\approx\dfrac{2n^2}{3n^2}=\underline{\underline{\dfrac{2}{3}}}\quad\Rightarrow\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{2}{3}}=1.$

Sejtjük a határértéket.

<u>Tört növelése:</u> i) $2n^2-3n+1<2n^2+1n^2=\underline{\underline{3n^2}}$ számláló.

ii) $3n^2+11>3n^2$

így $\dfrac{2n^2-3n+1}{3n^2+11}<\dfrac{3n^2}{3n^2}=1.$

Tört csökkentése:

i) $2n^2-3n+1 > 2n^2-3n > 2n^2-\frac{1}{3}n^2 = \underline{\frac{5}{3}n^2}$

$\Downarrow$

$3n < \frac{1}{3}n^2$

$9n < n^2$

$9 < n$

ii) $3n^2+11 < 3n^2+11n^2 = 14n^2$.

Így: $\frac{n^2-3n+1}{3n^2+11} > \frac{\frac{5}{3}\cancel{n^2}}{14\cancel{n^2}} = \frac{5}{3}\cdot\frac{1}{14} = \underline{\frac{5}{42}}$

$$\sqrt[n]{1} > \sqrt[n]{\frac{n^2-3n+1}{3n^2+11}} > \sqrt[n]{\frac{5}{42}}$$

Ha $n\to\infty$: $\sqrt[n]{1} \to 1$, $\sqrt[n]{\frac{5}{42}} \to 1$, $\sqrt[n]{\frac{n^2-3n+1}{3n^2+11}} \to \underline{\underline{1}}$.

2/O $\lim\limits_{n\to\infty}\left(\underbrace{\frac{n+3}{n-5}}\right)^{n^2} = ?$

$\to 1$, ha $n\to\infty$.

„e"-csavar?

$$\lim_{n\to\infty}\left(\frac{n+3}{n-5}\right)^{n^2} = \lim_{n\to\infty}\left(\frac{n-5+8}{n-5}\right)^{n^2} = \lim_{n\to\infty}\left(1+\frac{8}{n-5}\right)^{n^2} =$$

$$= \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{\frac{n-5}{8}}\right)^{\frac{n-5}{8}}}_{\to e,\ \text{ha } n\to\infty}\right]^{\underbrace{\frac{8}{n-5}\cdot n^2}_{\to\infty \text{ Ha } n\to\infty}} = \text{„}e^{\infty}\text{"} = \underline{\underline{\infty}}.$$

2/P $\lim\limits_{n\to\infty}\sqrt[n]{\frac{2n^3+1}{n^2-3n}} = ?$

Ha n elég nagy: $\frac{2n^3+1}{n^2-3n} \approx \frac{2n^3}{n^2} = 2n$.

Így $\sqrt[n]{2n} = \sqrt[n]{2}\,\sqrt[n]{n} \Rightarrow 1$ Ha $n\to\infty$.

RENDŐRELV KELL

**

** folyt

Felső becslés: i) $2n^3+1 < 2n^3+n^3 = 3n^3.$

ii) $n^2-3n > n^2 - \frac{1}{3}n^2 = \frac{2}{3}n^2.$

$\Uparrow$

$3n < \frac{1}{3}n^2$

$9n < n^2$

$n > 9.$

Így: $\frac{2n^3+1}{n^2-3n} < \frac{3n^3}{\frac{2}{3}\cancel{n^2}} = 3n \cdot \frac{3}{2} = \frac{6}{2}n.$

Alsó becslés:

i) $2n^3+1 > 2n^3.$

ii) $n^2-3n < n^2.$

Így: $\frac{2n^3+1}{n^2-3n} > \frac{2n^3}{n^2} = 2n.$

$$\sqrt[n]{\frac{6}{2}n} > \sqrt[n]{\frac{2n^3+1}{n^2-3n}} > \sqrt[n]{2n}$$

Ha $n \to \infty$: $\underbrace{\sqrt[n]{\frac{6}{2}}\sqrt[n]{n}}_{\downarrow\ 1.}$ $\qquad \sqrt[n]{\frac{2n^3+1}{n^2-3n}} \to 1 \qquad \underbrace{\sqrt[n]{2}\sqrt[n]{n}}_{\downarrow\ 1}$

$$a^4-b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$$

2/a)

$$\lim_{n\to\infty} \frac{\sqrt[n]{256}-1}{\sqrt[n]{4}-1} = \lim_{n\to\infty} \frac{\sqrt[n]{4^4}-1}{\sqrt[n]{4}-1} = \lim_{n\to\infty} \frac{\left(\sqrt[n]{4}\right)^4-1}{\sqrt[n]{4}-1} =$$

$$= \lim_{n\to\infty} \underbrace{\left(\sqrt[n]{4}\right)^3}_{\downarrow\ 1} + \underbrace{\left(\sqrt[n]{4}\right)^2}_{\downarrow\ 1} + \underbrace{\left(\sqrt[n]{4}\right)^1}_{\downarrow\ 1} + 1 = \underline{\underline{4}}$$