Lab Assignment #4

Fourier Transform

By: Prof. Dr. rer. nat. habil. Albrecht Rost
University of Applied Sciences Merseburg
Department 1: Computer Science and Applied Natural Sciences
Physics, Laboratory of Measurement Technology
e-mail: albrecht.rost@in.fh-merseburg.de

Objective

- Measure different time functions, calculate the spectral functions using FFT and compare these with theoretical results
- Determine the transfer function of a first-order low pass from the comparison of the spectra of the output and input voltages
- Determine resonant frequency and loss constant of a parallel resonant circuit from the measured time function and the equivalent spectral function

Equipment

- Agilent 54622A Deep Memory Oscilloscope or Agilent 54600B Oscilloscope
- Agilent 33120A Function Generator
- Agilent VEE software

A. Theoretical Introduction

Many physical processes happen over time. Therefore, in many cases, the result of investigation of physical system is a time-dependent function. On the other hand, using a sinusoid signal with constant amplitude and variable frequency to stimulate a system and measuring amplitude and phase angle of the response signal, a frequency-dependent function will be obtained. Also, we have two possibilities to describe the behaviour of a physical system: we can do it in the time domain or in the frequency domain. Changing the description of a system can be done – obeying definite restrictive requirements - using the FOURIER transform or the inverse FOURIER transform.

The FOURIER transform assigns a frequency function $f(\omega)$, the so-called spectral function, to a time function F(t). If the time function is periodically and unlimited in the time domain, i.e., if

$$F(t) = F(t+T),$$

where T denotes the period, the function F(t) can be developed into an infinite series of sine and cosine terms

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n w_0 t + b_n \sin n w_0 t)$$

with the FOURIER coefficients

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} F(t) \cos n w_0 t \, dt,$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} F(t)\sin n w_0\, t dt.$$

The constant term $a_0/2$ is equal to the mean value $\overline{F}$ of the function F(t). If the function F(t) is described by an analytical expression the FOURIER coefficients a_n and b_n can be calculated. The term with the frequency $\omega_0 = 2\pi/T$ is the so-called first harmonic (basic oscillation). All other terms, the frequencies of which are integer multiples of the basic frequency, are called higher harmonics.

The determination of the FOURIER coefficients of a given time function is called FOURIER analysis. The given function F(t) can be described unequivocally using the FOURIER coefficients. On the other hand, we speak about FOURIER synthesis, if we calculate a periodic time function from a given set of harmonics.

In many practical applications time functions are used which are not necessarily periodically but time-limited. In this case FOURIER's theorem can not be used. The connection between the time-limited function and the spectral function can be described by the FOURIER integral (FOURIER transform)

$$F(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} \underline{c}(w)\, e^{jwt} dw$$

and the inverse FOURIER transform

$$\underline{c}(w) = \int_{-\infty}^{+\infty} F(t)\, e^{-jwt}\, dt\ ,$$

respectively.

Using state-of-the-art algorithms these transformations can be done in real time (FFT: fast FOURIER transform; iFFT inverse fast FOURIER transform), which opens new possibilities of processing and interpretation of measurement data. But it must be taken into account that in this case discrete functions are used with a well-defined number of data points and the transformation is done numerically. Therefore, all functions in the time domain as well as in the frequency domain are limited, and two sets of parameters exist: the time window t_m and the time step Δt in the time domain and the frequency window f_m and the frequency step Δf in the frequency domain, which determines the number of data points $N_t = (t_m/\Delta t) + 1$ or $N_f = (f_m/\Delta f) + 1$; t_m and f_m depend on one another:

$$\Delta f \sim \frac{1}{t_m}\ ,$$

$$f_m \sim \frac{1}{\Delta t}\ .$$

In practice, it is necessary to optimize these parameters taking into account the sampling theorem (see also exercise *Sampling Theorem / Lab Assignment #3).* Mostly, a very high oversampling will be used to obtain a good representation of the measured functions in the time domain as well as in the frequency domain.

B. Simulation Experiments

For better understanding of theory and usage of FOURIER transform you should study some simulation programs. To run these programs you need the run-time version of Agilent VEE (Version 5). With these programs you can investigate the properties of the FOURIER transform applied to different time functions, which often are used in measurements, as well as different aspects of their application in the measurement technology.

1. FFT1.VXE

With this program you can study the spectral functions of different periodic standard time functions, which often are used in the measurement technology. Notice that discrete data processing will be used. Therefore, the properties of the digitalized time function depends on two characteristic parameters: the time window t_m (the duration of measurement), adjusted by the number of periods and the time step Δt. The equivalent parameters in the frequency domain – the frequency step Δf and the frequency window f_m - are inversely proportional to the parameters of the time domain.

Investigate the spectral function of different time functions.

Observe of the influence of parameters of the time domain on the parameters of the frequency domain; look at the bad reconstruction of functions, if only a little oversampling is used.

2. FFT2.VXE

The input signal consists of one or two damped sinusoidal functions. You can adjust the amplitude A_i, the frequency f_0 and the loss factor d of the signals (for only one signal you must adjust $A_2 = 0$).

Investigate the FOURIER transform with different input signals.

Note: The spectral function of a damped oscillation is the resonant curve, from which the characteristic parameters f_0 and d can be determined (see page 4).

3. FFT3.VXE

This program calculates the complex transfer function of a first-order low-pass with given cut-off frequency ω_c and displays it in a Bode diagram (magnitude and phase angle versus logarithm of frequency, being the measured result using a sinusoid input voltage with constant amplitude and variable frequency). The transformation into the time domain using iFFT gets a decreasing time pulse, the response function of a DIRAC pulse. (The DIRAC function is a pulse with a duration $t_p \to 0$ and an amplitude $A_p \to \infty$ corresponding to a constant spectral density.)

Investigate the influence of the cut-off frequency on the duration t_p of the output time function. The result agrees with an important basic law of transmission technology:

$$\omega_c \sim \frac{1}{t_p},$$

i.e., the bandwidth of a signal is inversely proportional to its duration.

4. FFT4.VXE

This program calculates a damped oscillation with adjustable values of the variables amplitude A, frequency f_0 and loss factor d as well as of the parameters time window t_m and time step Δt. From this time function the complex spectral function will be calculated and its magnitude as well as its real and imaginary part versus frequency will be displayed. This means that, the real time function contains all information's about the complex spectral function.

Note:

- The number of data points is different in the time and in the frequency domain.
- The inverse transformation of the complex spectral function results the original time function.

FFT5.VXE

This program calculates the complex spectral function of a damped oscillation with adjustable values of the variables amplitude A, frequency f_0 and quality Q and displays its magnitude and phase angle as well as its real and imaginary part versus frequency. From this complex spectral function the time function will be calculated and displayed.

Note: It is necessary to use the complex spectral function to obtain a correct inverse FOURIER transform.

A very important consequence for the measurement technology is: To obtain all information about a vibrating system it is advantageous to measure in the time domain because, the measurement in the frequency domain is very difficult and time-consuming.

C. Laboratory Experiments

1. Exercises

1.1. Functions in the time and the frequency domain

Generate different time functions using a function generator and an one-way rectifier. Measure the digitalized time functions using the computer and calculate the spectral functions using FFT. Compare the measured results with theoretical ones.

1.2. Transfer function of a low-pass

Determine the transfer function of a low pass by the distortion of the transmitted square wave signal comparing the spectra of the output and input signal.

1.3. Properties of a parallel resonant circuit

Display the damped oscillation of a parallel circuit using an oscilloscope and determine resonant frequency and loss constant from the oscillogram.

Measure the digitalized time signal and calculate the spectral function with FFT; determine resonant frequency and loss constant.

Investigate the influence of a parallel resistor.

Compare the results of the different methods.

2. Theoretical Background

The knowledge of the spectrum of standard time functions is important for many applications of these signals. A simple example is the rectification of different periodic signals. The mean value of periodic functions (without offset) is zero, but after rectification a non-zero mean value will be obtained (see also exercise *Mean Value of Periodic time Functions / Lab Assignment #1*).

A basic problem of the transmission technology is the measurement of the complex transfer function

$$\underline{g}(\omega) = \frac{\underline{F}_2}{\underline{F}_1}$$

($\underline{F}_1, \underline{F}_2$ input and output signals) using a sinusoid input signal with constant amplitude and variable frequency and measuring the amplitude and phase angle of the output signal. But this method is time-consuming and, especially, the exact measurement of the phase angle is difficult. An other possibility consists of the application of an input signal with a well-defined spectrum (e.g. a square wave) connected with the FOURIER transform. Than the magnitude of the transfer function can be obtained comparing the spectra of the input and the output signal.

A special technical problem is the investigation of the properties of a damped oscillating system. The measurement of the resonant curve is difficult and time-consuming. A very quick method is the measurement of the time function

$$F(t) = \hat{F} \cdot e^{-\delta t} \cdot e^{j\omega_0 t}$$

of the damped oscillation using a digitizing oscilloscope or a transient recorder. The resonant frequency f_0 can be determined directly from the displayed time function, and from the sequence of amplitudes F_n and F_{n+1} we obtain the logarithmic decrement

$$\ln \frac{F_n}{F_{n+1}} = \delta T = \frac{2\pi\delta}{\omega_0}$$

and, finally, the loss constant

$$\delta = \frac{1}{T} \cdot \ln \frac{\hat{F}_n}{\hat{F}_{n+1}} = f_0 \cdot \ln \frac{\hat{F}_n}{\hat{F}_{n+1}} \quad .$$

Note: A higher accuracy will be achieved, if the special sequence F_n and F_{n+i} with

$$\ln \frac{F_n}{F_{n+i}} = 1$$

is used. With

$$\ln \frac{\hat{F}_n}{\hat{F}_{n+i}} = i \cdot \ln \frac{\hat{F}_n}{\hat{F}_{n+1}}$$

we obtain

$$\delta = \frac{1}{i \cdot T} = \frac{f_0}{i} \quad .$$

Applying the FOURIER transform on this time function the resonance curve of the system will be obtained. In the case of small loss ($\delta << \omega_0$) the resonant frequency corresponds to the maximum of this curve and the loss constant is given by

$$\delta = \frac{\omega_2 - \omega_1}{2} = \frac{\omega_0}{2Q} = \omega_0 \frac{d}{2}$$

($\omega_2 - \omega_1 = B$ bandwidth, measured at $1/\sqrt{2}$ of the maximum value of the resonant curve; Q quality; d loss factor). This method is advantageous especially in the case of systems with many resonance frequencies.

3. Instructions to the Laboratory Exercises

The measurements are done using a special ADC connected with the computer via an IEEE488 interface and the Agilent VEE program *TRANSFFT*. Using this program the ADC works as a transient recorder with a memory of 2 kByte and a sampling rate till 300 kHz.

1.1.

In this exercise the spectral functions (FOURIER coefficients) of periodic standard functions are to be determined. These are often used for measuring transfer functions. Use the function generator 33120A and a simple one-way rectifier (diode: BAT 85; resistor: 1 kΩ) to generate different time-dependent voltages, digitalize and save the signals using the transient recorder and execute the FOURIER transform to determine the FOURIER coefficients. Note to the correct relation between input sensitivity and signal amplitude. Keep in mind the dependence between measurement duration and sampling rate and choose optimal values. Print out characteristic results and compare these with those from the literature.

Use an adder (see Fig.1; construct the circuit on the *hps* Analogue Development Board) to superpose two sinusoid signals U_{11} and U_{12} with a small frequency difference, digitalize and save the signal U_2 . Compare the results in time and frequency domain.

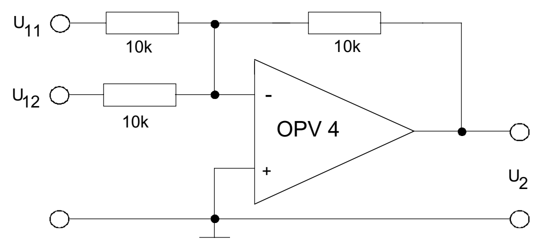

Fig. 1: Adder

1.2.

Determine the amount of the transfer function $g(\omega) = U_2 / U_1$ comparing the spectral function of the output voltage U_2 with that of the input voltage U_1 . Fig. 2 shows the principle of the measurement circuit. Use a square wave input voltage ($U = 5V, f = 500Hz$) and measure the amplitudes of the spectral lines of the input as well as of the output voltage carefully using the cursors on the screen. Draw the resulting transfer function $g(n \cdot f_0)$ in a Bode diagram and compare it with the theoretical function calculated with the resistor R and the capacity C.

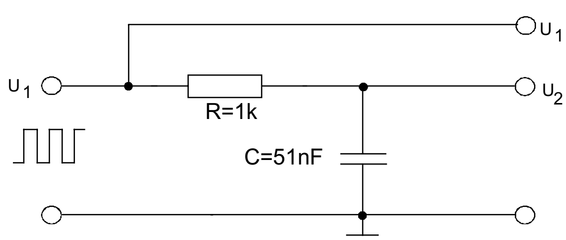

Fig. 2: RC low pass

1.3.

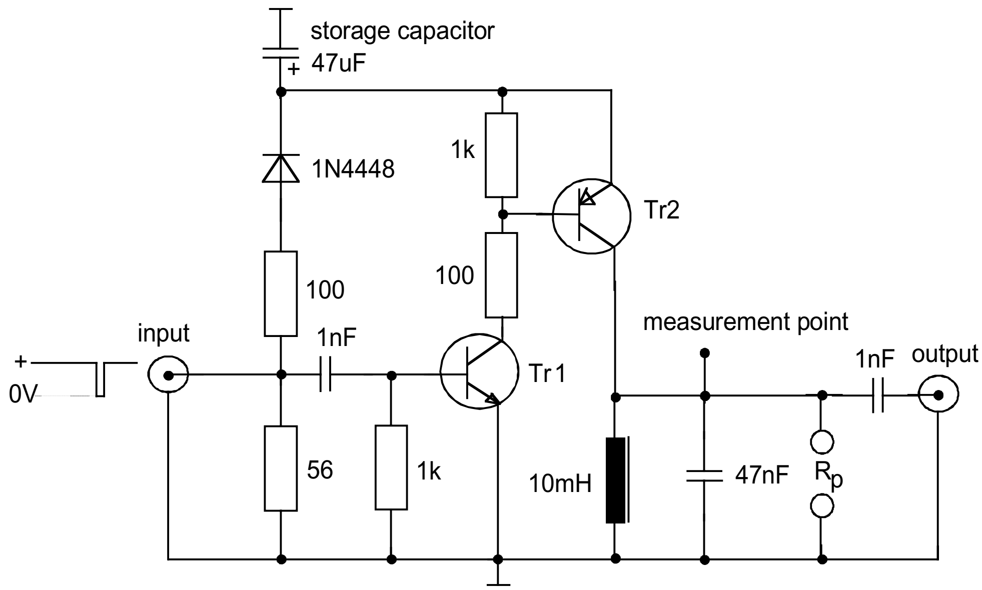

Fig. 3: Parallel resonant circuit (measurement schema)

The principle of the measurement is shown in Fig. 3. In the steady state the input voltage is +5V, and the storage capacitor is charged. For a short time t_P the input will be switched to zero and a damped oscillation will be generated.

At first display the damped oscillation using the oscilloscope 54600B and determine resonant frequency and damping constant from the oscillogram. Then measure the oscillation using the transient recorder, carry out the FOURIER transform and determine resonant frequency and loss constant from the spectral function. Compare the second result with those from the first experiment. Investigate the influence of a parallel resistor R_p.

Use two characteristic results with different values of the resistor R_p for your documentation.