Experiment No. 1.

Low-Pass Filters; Step Response vs. Q

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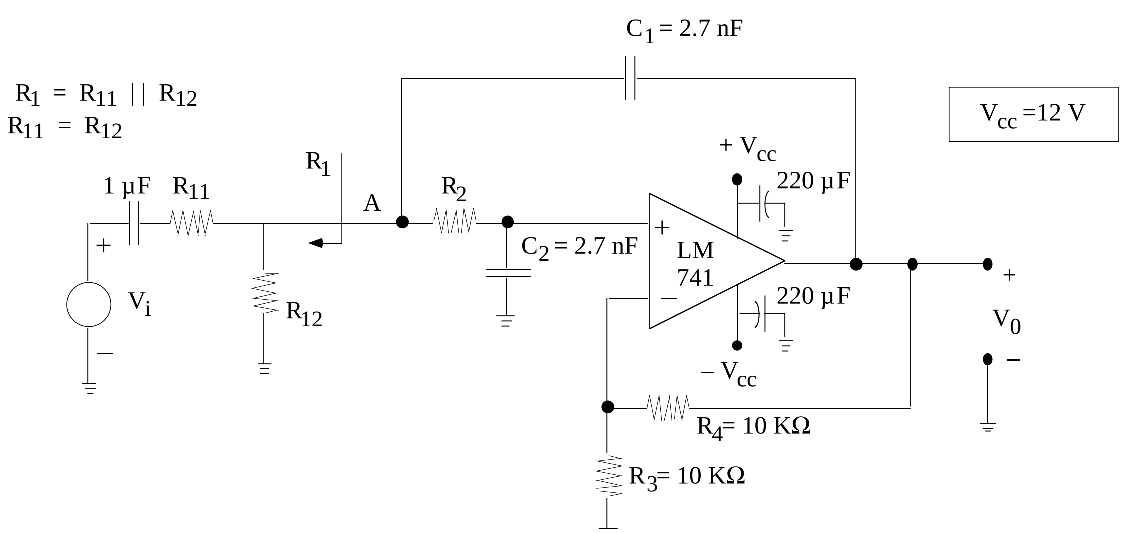
Purpose

To determine the frequency response of an active second order low-pass filter circuit for underdamped (Q > 0.5) and overdamped (Q < 0.5) cases, and to see the effect of Q in time domain using the step-response.

1.0 Low-Pass Filter Implementation:

Equipment: The whole Agilent rack.

A Sallen-Key non-inverting low-pass filter is shown below:



The transfer function is given by:

$$\frac{V_o}{V_i} = \frac{\frac{1}{R_1 R_2 C_1 C_2}\left(1+\frac{R_4}{R_3}\right)\left(\frac{R_{12}}{R_{12}+R_{11}}\right)}{s^2 + \left(\frac{1}{R_2 C_1} + \frac{1}{R_1 C_1} - \frac{R_4}{R_2 R_3 C_2}\right)s + \frac{1}{R_1 R_2 C_1 C_2}} \qquad (s = j\omega)$$

where $\left(\frac{R_{12}}{R_{12}+R_{11}}\right)$ is the input voltage divider, and $\left(1+\frac{R_4}{R_3}\right)$ is the non-inverting op-amp low-frequency gain when C1 and C_2 are open-circuited. The 1 µF capacitor at the input is a DC-block capacitor and will act as a short circuit for f > 100 Hz (EECS 210).

For $C_1 = C_2 = C$, we have:

$$K = \left(1+\frac{R_4}{R_3}\right)\left(\frac{R_{12}}{R_{11}+R_{12}}\right) \quad \textit{low frequency filter gain}$$

$$\omega_0 = \frac{1}{C\sqrt{R_1 R_2}} \quad \textit{cut-off frequency (also called corner frequency)}$$

$$R_2 = \frac{1}{2Q\omega_0 C}\left(1+\sqrt{1+4Q^2(K'-2)}\right) \qquad K' = \left[1+\frac{R_4}{R_3}\right] \qquad K' \geq 2$$

$$R_1 = \frac{1}{\omega_0^2 C^2 R_2}$$

and $$Q = \frac{1}{\sqrt{\frac{R_2}{R_1}}+\sqrt{\frac{R_1}{R_2}}-\frac{R_4}{R_3}\sqrt{\frac{R_1}{R_2}}} = \frac{1}{\sqrt{\frac{R_2}{R_1}}+\sqrt{\frac{R_1}{R_2}}\left(1-\frac{R_4}{R_3}\right)}$$

For the case of $R_4 = R_3$ and $R_{11} = R_{12}$, we have $K' = 2$ and:

$$K = 1, \quad \omega_0 = \frac{1}{C\sqrt{R_1 R_2}} \quad \textit{and} \quad Q = \sqrt{\frac{R_1}{R_2}}$$

In reality 0.95 < K < 1.5 due to resistance values inaccuracies (±5%).

Low-Q Active Filter:

1. Assemble the circuit as shown above with:

   $R_{11} = R_{12} = 2.4$ KΩ and $R_2 = 33$ KΩ

   $C_1 = C_2 = C = 2.7$ nF

   This is the low-Q case with Q ~ 0.19, f_o ~ 9.3 KHz. (Again, f_o can change by ±10% due to capacitor values inaccuracies of ±10%).

   As indicated by Problem 5 in the Pre-Lab, the transfer function has effectively 2-poles, one at $f_1 \simeq Q f_o$, and one at $f_2 \simeq f_o/Q$.

   ❑ Draw the filter circuit in your notebook.

2. ❑ Measure the op-amp DC voltages, V(-), V(+) and V_o and make sure that they are all in the mV levels. Write them in your notebook.

<u>Frequency Response</u>:

3. Connect V_o to scope channel 1 and V_i to channel 2.

   a. ❑ Set the function generator to give a sinusoidal voltage of Vppk = 1 V and measure the frequency response (V_o/V_i) from 50 Hz to 200 KHz (amplitude only). For the low-Q case, you should do it at 100, 200, 500, 1000 Hz, etc. Above 20 KHz, the output voltage will be very low and you should increase the input voltage to 10 V ppk in order to measure accurately V_o.

   ❑ b. Measure the phase delay between the input and output waveform in the region around 8-10 KHz and determine exactly the frequency where it is -90°. This is ω_o (= 2 π f_o)!

   ❑ c. Measure V_o/V_i at this frequency ($|H(\omega_o)|$ = K Q $\simeq$ Q for K $\simeq$ 1). This is your Q!

   (Remember, that under the Measure [Time] menu, you will find a softkey at the bottom of the screen which measures the <u>phase</u> delay between Channel 1 and Channel 2.)

<u>Step Response</u>:

4. ❑ Set the function generator to give a <u>square wave</u> voltage of f = 500 Hz and Vppk = 1 V. Measure the risetime of the output waveform (10% - 90% of peak value). Draw the waveform on your lab notebook.

   (Remember that under the Measure [Time] menu, you can find a <u>Risetime</u> softkey at the bottom of the screen.)

5. ❑ The risetime is dominated by the first pole at $f_1 = 1/2\pi C_2 R_2 \sim 1.8\,KHz$. Therefore C_1 has virtually no effect on this pole. Remove C_1 from the circuit and measure the risetime again. What do you notice?

6. ❑ Now, put C_1 back into the circuit. Change the input square-wave frequency to 4 KHz and measure $V_o$ppk. Plot V_i and V_o and label the ppk voltages. Why does V_o look like a triangular wave?

High-Q Active Filter:

1. Change the values of R_{11} and R_{12}, and R_2 to be:

   $R_{11} = R_{12} = 120$ KΩ and $R_2 = 1$ KΩ

   Keep $C_1 = C_2 = C = 2.7$ nF.

   This is the high Q case with $Q_{(ideal)} = \sqrt{R_1/R_2} = 7.7$. However, since C_1 is not identical to C_2 and also there are some parasitic capacitances in the circuit, you will find a Q between 3 and 12. The resonant frequency is:

   $f_o = \dfrac{1}{2\pi C\sqrt{R_1 R_2}} \sim 7.6\,KHz$ but could change (±10%) due to capacitor values inaccuracies.

   The low frequency gain, K, is still equal to 1 (0.95 < K < 1.05 due to resistance inaccuracies).

   ❑ Draw the circuit in your lab notebook.

Frequency Response:

2. a. ❏ Set the function generator to a sinewave with V_{ippk} = 1 V. Measure the frequency response from 50 Hz to 200 KHz.

Be careful, the frequency response changes very quickly around f_o! First determine f_o $\left(\max V_o/V_i\right)$ by a quick frequency scan (using the knob), and then measure the frequency response taking several more points around f_o. (For example; the -3 dB, -6 dB, -10 dB points of H(ω) max).

b. ❏ Measure the phase of V_o/V_i around f_o and determine the frequency where the phase delay is -90°. This frequency will be very close to where |H (ω)| is maximum and is the exact ω_o.

c. ❏ Measure V_o/V_i at this frequency. (|H (ω_o)| = K Q ≃ Q for K ≃ 1). This is your Q!

3. Set the function generator for a square wave of frequency f_o (whatever you have measured, ~ 7.6 KHz) and Vppk = 1 V.

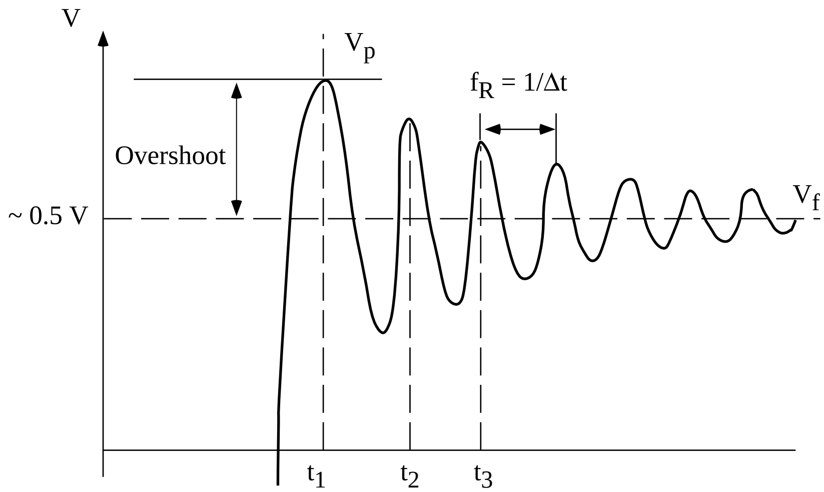
a. ❏ Measure the output voltage. Plot V_i and V_o and label the ppk voltages. What is the shape of the output voltage waveform?

b. ❏ Measure V_i and V_o in frequency domain in dB (f_o, $3f_o$, $5f_o$ and $7f_o$). Be careful, choose at least a 100 KHz frequency span so as not to get aliasing on the FFT.

Step Response:

4. ❏ Set the function generator to a 200 Hz square wave with Vppk = 1 V. Sketch the input and output voltage in your lab notebook for one period and label the output voltage when the ripple settles down, (V_f).

a. ❏ For the positive portion of the waveform, measure the exact time of the first peak (t_1) and its value (V_p). (You can do this using the Measure [Cursors] menu.) The overshoot value is (V_p - V_f).



b. ❏ From the waveform determine the ripple frequency, f_R. (Again, you can do this quickly under the Measure [Cursors] menu).

c. ❑ If you have a Q lower than 4-5, measure the values and times of the second, third and fourth peaks. You will need this to calculate Q in the post-lab report. If you have a high-Q (Q > 4-5), then measure the third, fifth and seventh peaks (value & time). This will result in a more accurate determination of Q.

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Pre-Lab Assignment

1. Using the Golden Rules (Ideal OP-Amp):
(Ignore the 1 µF DC-block capacitor. Assume it is a short-circuit at all frequencies.)

   a. In the low-pass filter, why is $R_1 = R_{11} \parallel R_{12}$? (Disconnect the circuit to the right of the arrow and determine the Thevenin's equivalent of the source circuit.)

   b. Draw the filter circuit at $\omega \ll \omega_o$ (caps. are open-circuit) and determine V_o/V_i.

   c. Draw the filter circuit at $\omega \gg \omega_o$ and determine V_o/V_i.

   d. What is the input impedance of the circuit (seen by V_i) for $\omega \ll \omega_o$?

2. a. Check that the resistor/capacitor values given in the experiment result in the quoted f_o, Q, K for the low-Q and high-Q cases.

   b. In the low-pass filter, the circuit will <u>never</u> work if R_{12} is not present (and V_s is a pure <u>ac</u> source). Explain why (think of the non-ideal op-amp properties).

3. The frequency response of a first order filter is given by:

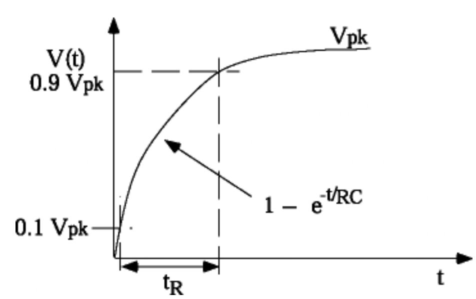
$$\frac{V_o}{V_i} = \frac{1}{\left(1+\dfrac{s}{\omega_1}\right)} \qquad \text{where } \omega_1 = {}^{1}\!/_{RC} \quad \text{and} \quad f_1 = \frac{1}{2\pi RC}$$

The risetime of a function (t_R) is defined as the time from 0.1 Vpk to 0.9 Vpk.

The time response to a step function of amplitude 1 is given by:

$$V_o(t) = 1 - e^{-t/RC} = 1 - e^{-t/\tau}$$

where $\tau = RC \equiv$ time constant.



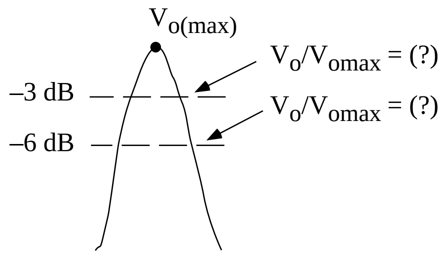
   a. Determine t_R as a function of τ (the time constant).

   b. Express t_R as a function of f_1.

(This expression is very useful, since once you know the risetime in seconds, you can quickly determine the corner frequency in Hz (and vice versa).)

4. Calculate $V_o/V_{o(max)}$ (for a constant V_i) when the filter response (transfer function) <u>drops</u> by: 3 dB (really –3 dB), 6 dB, 10 dB, 15 dB, 20 dB, 30 dB, 40 dB.

Graphically, this means



5. A low-pass filter transfer function is given by:

$$\frac{V_o}{V_s}(s) = H(s) = \frac{K\,\omega_o^2}{s^2 + (\omega_o/Q)s + \omega_o^2} \qquad s = j\omega \qquad f_o = 10\,KHz$$

and $$|H(\omega)| = \frac{K\,\omega_o^2}{\sqrt{\left[\omega_0^2 - \omega^2\right]^2 + \left[(\omega o/Q)\,\omega\right]^2}}$$ and $K \equiv$ low frequency gain

a. Derive |H(ω)|.

b. For K = 1 and using MATLAB, plot on the same graph (dB, log f) the filter response for Q = 0.2, Q = 1 and Q = 10. The horizontal scale should be from 0.01 f_O (100 Hz) to 100 f_O (1 MHz). The vertical scale should be from +20 dB to -60 dB. Reminder: dB = 20 log |H(ω)|.

c. Derive the equation of the phase of H(ω) and plot on the same graph the phase for Q = 0.2, Q = 1 and Q = 10 over the above mentioned frequency range. What is the phase of H(ω) as ω→0, ω→∞ and ω = ω_O.

d. Prove that at $\omega_o, \; |H(\omega)| = KQ$ and the phase of H(ω_O) is -90°.

e. For the special case of Q << 1 (Q = 0.2), the transfer function can be very well approximated by two poles at $\omega_1 = Q\omega_O$ and $\omega_2 = \omega_O/Q$. For Q << 1, ω_1 dominates the response. For the case of Q = 0.2, plot

$$H_1(s) = \frac{1}{\left(1 + \frac{s}{\omega_o Q}\right)\left(1 + \frac{s}{\omega_o/Q}\right)}$$

and compare it with the function of part (b) with Q = 0.2 (on the same graph).

f. Prove mathematically that, for Q << 1, H_1(s) is nearly equal to H(s). You can do this by expanding the denominator of H_1(s), determining which factor you must ignore so that H_1(s) = H(s), and proving that for Q << 1, this factor is insignificant.

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Lab Report Assignment

1. a. Draw the filter circuit and neatly summarize <u>all</u> your measured data for the low-Q and high-Q cases (R values, V^+, V^-, V_o (DC), f_o, Q, f_1, t_R, overshoot, t_1, etc. ..., everything except the frequency response data).

   b. Using Matlab, plot the measured transfer function for the low-Q, high-Q filters <u>on the same Bode-plot</u> (f: 50 Hz–200 KHz, dB: +10 or +20 dB–depending on your measured high-Q- to -60 dB minimum). Label clearly the low frequency gain (K), the "'resonant" frequency (f_o) and the Q of the filter for the low-Q and high-Q cases.

2. Low-Q Filter:

   a. From the bode-plot, determine the low frequency pole, f_1, (-3 dB pt). Knowing ω_o (or f_o) from the phase measurement and that $f_1 \simeq Qf_o$, determine Q.

   b. Using the risetime (t_R) from the time domain measurements, determine f_1 using the pre-lab problem #3. Determine Q (knowing ω_o). Does it agree with part (a) above?

   c. Knowing the low frequency pole, f_1, and for a square-wave of V_i = 1 V_{ppk} and f = 4 KHz, calculate V_{oppk} and compare with measurements of Section 6 of the low-Q design.

3. High-Q Filter:

   a. From the time domain measurements, determine f_o from the ripple frequency. Compare with frequency domain data.

   b. From the time domain measurements, determine Q from the overshoot value. Compare with frequency domain data.

   c. From the time domain measurements, determine Q from the ripple peaks. In one case, use the first two measured peaks (and times), and in another case, use the first and last measured peaks (and times). Compare both Q's with the frequency domain data.

4. a. For a square-wave of V_{pk} = 0.5V (V_{ppk} = 1 V) and f = f_o (~ 7.6 KHz–use your values), calculate the fundamental, third, fifth and seventh harmonic levels of the square-wave (in V and dB) <u>before</u> and <u>after</u> it passes by the high-Q low-pass filter. Compare with frequency-domain measurements for V_i and V_o. (Use Fourier-Series to calculate the frequency components of the input square-wave).

   b. For V_o, what is the value (in V and dB) of the third and fifth harmonic level compared to the fundamental? Is this a "clean" sinewave at f_o? Calculate the THD (total harmonic distortion) of the output signal (THD is defined in your EECS 210 lab manual).

Some Problems with the Active Sallen-Key Low-Pass Filter:

1. Oscillations:

The denominator of the transfer function $H(s) = V_o / V_i$ is:

$$s^2 + \left(\frac{1}{R_2 C_1} + \frac{1}{R_1 C_1} - \frac{R_4}{R_2 R_3 C_2} \right) s + \frac{1}{R_1 R_2 C_1 C_2}$$

When $\left(\frac{R_4}{R_2 R_3 C_2} \right) > \left(\frac{1}{R_2 C_1} + \frac{1}{R_1 C_1} \right)$, the denominator has the form:

$$s^2 - a_1 s + a_o = 0$$

The solutions of this quadratic equation will always result in at least one positive pole, and therefore the system response will "blow" up. Since this is an active circuit, the filter will oscillate and generate a somewhat sinusoidal voltage limited by $\pm$Vcc. Therefore, always choose the components such that $\frac{R_4}{R_2 R_3 C_2} < \frac{1}{R_2 C_1} + \frac{1}{R_1 C_1}$ and you are guaranteed a good second-order low-pass filter. The above is especially true for high-Q filters where $R_1 >> R_2$. In this case, oscillations will not occur when:

$$\frac{R_4}{\cancel{R_2} R_3 \cancel{C_2}} < \frac{1}{\cancel{R_2 C_1}} + \cancel{\frac{1}{R_1 C_1}}\ \textit{negligible} \qquad (C_1 = C_2 = C)$$

$$\Rightarrow \frac{R_4}{R_3} < 1 .$$

Therefore, always maintain $R_4 < R_3$ for *stable* high-Q filters.

2. Uncertainty in Q for high-Q designs:

The Q of a Sallen-Key non-inverting low-pass filter (with $C_1 = C_2 = C$) is given by:

$$Q = \frac{1}{\sqrt{\frac{R_2}{R_1}} + \sqrt{\frac{R_1}{R_2}} \left(1 - \frac{R_4}{R_3} \right)}$$

which simplifiers to $Q = \sqrt{R_1 / R_2}$ for $R_4 = R_3$.

However, in a high-Q design, $R_1 >> R_2$ and any slight non-equalities in R_4 and R_3 can change the value of Q by a large fraction. Let us consider the example where R_1 = 30 KΩ, R_2 = 1 KΩ, and:

$$Q = \frac{1}{0.183 + 5.48 (1 - R_4 / R_3)}$$

which simplifies to Q = 5.5 for $R_4 = R_3$ exactly.

However, for $R_4 = 1.02\ R_3$, we have Q = 13.6, and for $R_4 = 0.98\ R_3$, we have Q = 3.4. Therefore, 3.4 < Q < 13.6, for a $\pm$2% change in the R_4/R_3 resistor ratio!

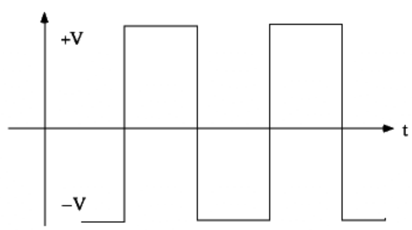
This holds true for $C_1 = C_2 = C$. However, $C_1 \neq C_2$ due to capacitor inaccuracies and parasitic capacitances in the circuit. Therefore, even if one maintains $R_4 = R_3$, Q can still vary between 3 and 14 due to capacitance variations alone. Therefore, the Sallen-Key filter is simply not a good design for high-Q filters!

In the above example, one could also ask what happens to Q when $R_4/R_3 = 1.05$ (with $C_1 = C_2 = C$). In this case, Q < 0 and the circuit will oscillate (see above). Therefore, it is always good to maintain $R_4 \leq R_3$ and use $\pm$ 1% resistors (instead of the standard $\pm$5% resistors) for R_4 and R_3 in Sallen-Key high-Q filters.

For the above reasons, a sensitivity analysis is always done in commercial engineering applications. The sensitivity analysis tells the engineer how much variation in output specifications to expect for a small change in component value. This is especially important in high-volume manufacturing, where a parts vendor could supply you with a whole run of components, all at the upper end of their tolerance limit."

Equations you may need for the lab report:

1) The Fourier-Series of a square-wave signal is given by $V(t) = \sum_{n=1}^{\infty} \frac{4V}{n\pi} \sin\left(n\omega_o t\right)$ where V is the peak voltage (and not peak-to-peak).



2) Total Harmonic Distortion, THD, is defined as:

$$\mathrm{THD}(\%) = \sqrt{\frac{\sum\left(P_{\mathrm{harmonics}}\right)}{P_{\mathrm{signal}}}}\ \mathrm{x}\ 100$$

where

$$\sum P_{harmonics} = \left(\frac{V_{2f_0}^2}{R_L}\right) + \left(\frac{V_{3f_0}^2}{R_L}\right) + \left(\frac{V_{4f_0}^2}{R_L}\right) + \left(\frac{V_{5f_0}^2}{R_L}\right) + \ldots \qquad \underline{\mathrm{V\ in\ rms!}}$$

$$P_{signal} = \left(\frac{V_{f_0}^2}{R_L}\right)$$