Newton-Raphson Method in Nonlinear Magnetics

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Abstract: The appropriate combination of the graphical user interface and functions of COMSOL Multiphysics and Matlab can be used efficiently to solve nonlinear problems in electromagnetics. The model of the problem to be solved and the weak formulations of the potential formulations obtained from Maxwell's equations can be implemented in the Physics menu of COMSOL Multiphysics, however the solver and the nonlinear hysteresis characteristics of the materials can be realized in the frame of Matlab. This procedure is shown in this paper through presenting three workshop problems.

Keywords: Nonlinear static magnetic field, Fixed point method, Newton-Raphson technique.

1. Introduction

Efficient implementation of nonlinear characteristics in electromagnetic field simulation environments is still an open question today. Problems including nonlinear characteristics can only be solved by iterative techniques, when a linearized problem is solved at every iteration step. The most widely used procedure in computational electromagnetism is the Finite Element Method (FEM). The unknowns are associated to the nodes or edges of the mesh. The number of unknowns can be very large, that is why a computationally efficient nonlinear solver must be worked out [1,2,3].

The nonlinear model has to simulate the hysteretic behavior as accurately as possible. At the same time it should be realized by a fast algorithm from the computational point of view, since the model is requested many times during the iteration. The model should handle nonphysical values of the magnetic field, too (e.g. in the vicinity of singularities).

The nonlinear characteristics can be handled by the polarization technique, when the magnetic field intensity or the magnetic flux density is split into a linear and a nonlinear part. The nonlinear part is depending on the constitutive relationship, and it is determined iteratively.

The most usual techniques are the fixed point method and the Newton-Raphson method. The fixed point algorithm can result in an unconditionally convergent method, which is, however very slow. The Newton-Raphson method is known as a fast iteration scheme, especially when the iterated solution is close to the true one. The behavior of its convergence is sometimes very strange.

2. Governing Equations

Maxwell's equations of the static magnetic field problems are as follows (Figure 1) [1,2,3]:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}, \; \nabla \cdot \boldsymbol{B} = 0, \; \text{in} \quad \Omega_0 \cup \Omega_m, \; (1)$$

where $\boldsymbol{H}$, $\boldsymbol{B}$ and $\boldsymbol{J}$ are the magnetic field intensity, the magnetic flux density and the source current density of the excitation coil, which is supposed to be known here. The constitutive relation between the magnetic field intensity and the magnetic flux density in the air region Ω_0 is $\boldsymbol{H} = \nu_0 \boldsymbol{B}$, where $\nu_0 = 1/\mu_0$ is the inverse of permeability of vacuum. The nonlinear constitutive relationship is represented by a nonlinear model in Ω_m. According to the polarization technique, the magnetic flux density and the magnetic field intensity are split into two parts as

$$\boldsymbol{B} = \mu\boldsymbol{H} + \boldsymbol{R}, \; \text{or} \; \boldsymbol{H} = \nu\boldsymbol{B} + \boldsymbol{I} \; \text{in} \; \Omega_m. \; (2)$$

Here μ, and ν are permeability-like, and reluctivity-like quantities, moreover $\boldsymbol{R}$ and $\boldsymbol{I}$ are the nonlinear terms depending on the input-output state of the applied hysteresis model.

The problem region is surrounded by the boundary $\Gamma_H \cup \Gamma_B$. On Γ_H, the tangential component of $\boldsymbol{H}$, on Γ_B, the normal component of $\boldsymbol{B}$ is set to zero, respectively, i.e. $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}$, on Γ_H, and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$, on Γ_B.
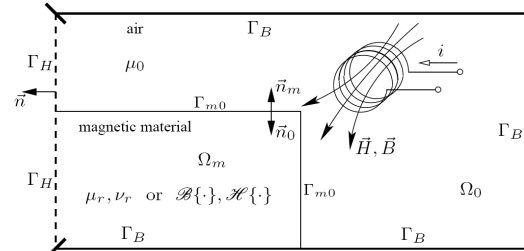

Figure 1. Scheme of static magnetic field problems.

2.1 The Magnetic Vector Potential

From $\nabla \cdot \boldsymbol{B} = 0$ in (1), the magnetic vector potential A can be introduced as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} . \qquad (3)$$

Substituting this relation back to (1) via (2), the nonlinear partial differential equation

$$\nabla \times [\nu \nabla \times \boldsymbol{A}] = \boldsymbol{J} - \nabla \times \boldsymbol{I} \qquad (4)$$

can be obtained, and the boundary conditions of the problem can be formulated as

$$[\nu \nabla \times \boldsymbol{A} + \boldsymbol{I}] \times \boldsymbol{n} = \boldsymbol{0}, \text{ and } \boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0} \qquad (5)$$

on Γ_H, and on Γ_B, respectively. The divergence of the magnetic vector potential can be selected according to Coulomb gauge, i.e. $\nabla \cdot \boldsymbol{A} = 0$. After some manipulations, Coulomb gauge can be enforced as

$$\nabla \times [\nu \nabla \times \boldsymbol{A}] - \nabla [\nu \nabla \cdot \boldsymbol{A}] = \boldsymbol{J} - \nabla \times \boldsymbol{I} . \qquad (6)$$

This results in two more conditions beyond (5), $\boldsymbol{A} \cdot \boldsymbol{n} = 0$, on Γ_H, and $\nu \nabla \cdot \boldsymbol{A} = 0$, on Γ_B. In this case A is approximated by nodal FEM.

The partial differential equation (4) with boundary conditions in (5) can be used when approximating A by vector FEM. In this case, $\boldsymbol{J}$ is approximated by the impressed current vector potential $\boldsymbol{T}$, since $\boldsymbol{J} = \nabla \times \boldsymbol{T}$, i.e. (4) has to be changed as

$$\nabla \times [\nu \nabla \times \boldsymbol{A}] = \nabla \times \boldsymbol{T} - \nabla \times \boldsymbol{I} . \qquad (7)$$

2.2 The Reduced Magnetic Scalar Potential

The source current density $\boldsymbol{J}$ is approximated by using the impressed current vector potential $\boldsymbol{T}$, and from the first Maxwell's equation

$$\boldsymbol{H} = \boldsymbol{T} - \nabla \Phi \qquad (8)$$

can be obtained, where Φ is called the reduced magnetic scalar potential. Substituting it back to (1) and to (2) results in the following nonlinear partial differential equation:

$$-\nabla \cdot (\mu \nabla \Phi) = -\nabla \cdot (\mu \boldsymbol{T}) - \nabla \cdot \boldsymbol{R} . \qquad (9)$$

On the boundary Γ_H, $\Phi = 0$ can be used, and $\mu(\boldsymbol{T} - \nabla \Phi) \cdot \boldsymbol{n} + \boldsymbol{R} \cdot \boldsymbol{n} = -b$ must be prescribed on the rest Γ_B.

Here $\boldsymbol{T}$ is approximated by edge finite elements, and Φ by nodal ones.

2.3 The impressed vector potential

This vector potential can be calculated by edge FEM and the following equation:

$$\nabla \times \nabla \times \boldsymbol{T} = \nabla \times \boldsymbol{J} \qquad (10)$$

with the boundary conditions $\boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0}$, on Γ_H, and $\boldsymbol{T} \cdot \boldsymbol{n} = 0$, on Γ_B.

3. Methods

The weak formulation of the nonlinear partial differential equations (6), (7) or (9) with the appropriate boundary conditions can be obtained by using the weighted residual method and some mathematical manipulations on them. The weak equations are not presented here, but they can be found in the references [1,2]. Here only the COMSOL implementation of the weak equations is presented. The weak form representation in the framework of COMSOL Multiphysics is very powerful and well applicable, all the weak equations can be implemented easily.

The potentials represented by nodal elements (i.e. Φ and the three components of nodal A) have to be realized by the `emes` (AC/DC Module -Statics - Electrostatics) application mode, and the potentials represented by edge elements (i.e. $\boldsymbol{T}$ and vector A) have to be formulated by the `emqa` (AC/DC Module-Statics-Magnetostatics) application mode, where the dependent variable `psi` is not used, but it should not be cleared, only the gauging has to set off. The other variables, like μ, ν, $\boldsymbol{I}$ (I_x, I_y, I_z), $\boldsymbol{R}$ (R_x, R_y, R_z) are usually represented by `emes` application mode through the `Multiphysics/Model Navigator` window. This means that some different physics modes represent one problem. One application mode represents one variable, which is not necessarily valid in the whole region, e.g. R_x, R_y, and R_z are valid only in the domain of nonlinear magnetic materials. Every physics mode must be checked through the `Physics/Subdomain Settings` window, where the checkbox `Active in this domain` must be checked in or out. The `equation system form` must set to `weak` in the window `Physics/Model Settings`, because the weak formulations of the partial differential equations of section 2 are implemented. In this case, the `Physics/ Equation System/ Subdomain Settings` and `Physics/ Equation System/ Boundary Settings` are used to insert weak equations and boundary conditions.

Constants, like the source current density, vacuum permeability, and so on can be defined in the window `Options/Constants`. The three orthogonal components of source current, i.e. J_x, J_y, and J_z can be defined in `Options/Expressions/ Subdomain Expressions`.

3.1. The Fixed Point Method

In the case of nodal representation of *A*, two different vector potentials are used. One in the ferromagnetic region (A1x, A1y, A1z), and one in the air region (A2x, A2y, A2z). It results in six emes application modes. The tangential component of them has to set to be continuous along iron/air interfaces. The other variables have to implement by using emes modes as well.

The following expressions must be typed when applying the magnetic vector potential:
```
nu0*
((A2zy-A2yz)*(A2zy_test-
A2yz_test)
+(A2xz-A2zx)*(A2xz_test-
A2zx_test)
+(A2yx-A2xy)*(A2yx_test-
A2xy_test))
+nu0*(A2xx+A2yy+A2zz)
*(A2xx_test+A2yy_test+A2zz_test)
-Jx*A2x_test
-Jy*A2y_test
-Jz*A2z_test,
```
and
```
nu1*
((A1zy-A1yz)*(A1zy_test-
A1yz_test)
+(A1xz-A1zx)*(A1xz_test-
A1zx_test)
+(A1yx-A1xy)*(A1yx_test-
A1xy_test))
+nu1*(A1xx+A1yy+A1zz)
*(A1xx_test+A1yy_test+A1zz_test)
+I1x*(A1zy_test-A1yz_test)
+I1y*(A1xz_test-A1zx_test)
+I1z*(A1yx_test-A1xy_test),
```
where nu1, I1x, I1y, and I1z are the fixed point coefficient in (2) and the nonlinear residual term. The terms according to the weak formulation can be seen by comparing these equations and the weak formulation.

The three components of the magnetic flux density can be calculated by Azy-Ayz, Axz-Azx, Ayx-Axy according to the three components of the curl operator. The variables Jx, Jy, and Jz are valid only in the region of coil.

The condition $A \cdot n = 0$ can be derived by the equation
```
-Ax*nx_emes-Ay*ny_emes
-Az*nz_emes
```
as constraint, and the same argument must be set in the test() function of the constrf tab.

The following three lines have to put as constraint if $n \times A = 0$ is prescribed:
```
-(ny_emes4*A2z-nz_emes4*A2y)
-(nz_emes4*A2x-nx_emes4*A2z)
-(nx_emes4*A2y-ny_emes4*A2x),
```
if emes4 denotes the application of A2x.

Along the interface between air and iron, $n \times A$ must be set to be continuous by the following three constraints:
```
-((ny_emes4*A2z-nz_emes4*A2y)-
(ny_emes4*A1z-nz_emes4*A1y))
-((nz_emes4*A2x-nx_emes4*A2z)-
(nz_emes4*A1x-nx_emes4*A1z))
-((nx_emes4*A2y-ny_emes4*A2x)-
(nx_emes4*A1y-ny_emes4*A1x)).
```

In case of vector representation of the magnetic vector potential, the following equation must be inserted in the weak tab of the Equation system window:
```
nu1*
((Azy-Ayz)*(Azy_test-Ayz_test)
+(Axz-Azx)*(Axz_test-Azx_test)
+(Ayx-Axy)*(Ayx_test-Axy_test))
-C*Tx*(Azy_test-Ayz_test)
-C*Ty*(Axz_test-Azx_test)
-C*Tz*(Ayx_test-Axy_test)
+I1x*(Azy_test-Ayz_test)
+I1y*(Axz_test-Azx_test)
+I1z*(Ayx_test-Axy_test).
```
This is valid in the nonlinear region. In air nu0 must be used, and the last three terms are deleted. The constant C should be used, because the current vector potential is usually calculated by using the value of source current 1A.

The following three simple lines have to put as constraint if $n \times A = 0$ is prescribed: -tAx, -tAy, -tAz.

The following expressions must be typed when applying the reduced magnetic scalar potential denoted by FI:
```
mu0*(FIx_test*FIx
+FIy_test*FIy
+FIz_test*FIz)
-mu0*C*(FIx_test*Tx
+FIy_test*Ty
+FIz_test*Tz),
```
and
```
mu1*(FIx_test*FIx+FIy_test*FIy
+FIz_test*FIz)
-mu1*C*(FIx_test*Tx
+FIy_test*Ty
+FIz_test*Tz)
-(FIx_test*R1x
+FIy_test*R1y
+FIz_test*R1z)
```

in the air region and in the domain filled with nonlinear material, respectively. Here `mu0` is defined as constant, `Tx`, `Ty`, and `Tz` are the three components of the `emqa` module according to the impressed current vector potential, and `C` is constant, because `T0` has been calculated by using the current 1A. The variable `mu1` can be constant, which is the fixed point coefficient in (2), finally `R1x`, `R1y`, and `R1z` are the three components of the nonlinear residual term. The Dirichlet boundary condition on Γ_H can be realized by `-FI` and `test(-FI)` on the `constr` and `constrf` tabs of the `Boundary Settings` window. The three components of the magnetic field intensity can be calculated by `Tx-FIx`, `Ty-FIy`, and `Tz-FIz`.

It is noted that, $\boldsymbol{R}$ and $\boldsymbol{I}$ should be eliminated in the case of linear magnetostatics.

3.2. The Newton-Raphson Method

The weak formulations of the Newton-Raphson method can be implemented by appending some terms to the equations of the fixed point technique.

The weak equation of the magnetic vector potential represented by nodal elements is as follows:
```
nu1*
((A1zy-A1yz)*(dA1zy_test-
dA1yz_test)
+(A1xz-A1zx)*(dA1xz_test-
dA1zx_test)
+(A1yx-A1xy)*(dA1yx_test-
dA1xy_test))+
(nud1x*(dA1zy-dA1yz)
*(dA1zy_test-dA1yz_test)
+nud1y*(dA1xz-dA1zx)
*(dA1xz_test-dA1zx_test)
+nud1z*(dA1yx-dA1xy)
*(dA1yx_test-dA1xy_test))
+nu1*(A1xx+A1yy+A1zz)*(dA1xx_tes
t+dA1yy_test+dA1zz_test)
+nu1*(dA1xx+dA1yy+dA1zz)*(dA1xx_
test+dA1yy_test+dA1zz_test)
+I1x*(dA1zy_test-dA1yz_test)
+I1y*(dA1xz_test-dA1zx_test)
+I1z*(dA1yx_test-dA1xy_test).
```
Here `dAx`, `dAy`, and `dAz` are the three components of $\Delta\boldsymbol{A}$, which is the unknown of the equation, and $\boldsymbol{A}$ is the magnetic vector potential in the previous nonlinear step, i.e. $\boldsymbol{A}$ is updated by $\boldsymbol{A}+\Delta\boldsymbol{A}$. The variables `nud1x`, `nud1y`, and `nud1z` are the three components of the differential reluctivity, which are changing in

every iteration step. The equation according to the air region can be obtained by eliminating the terms according to nonlinearity.

The weak equation of the magnetic vector potential represented by edge elements is the following in the nonlinear region:
```
nu1*
((Azy-Ayz)*(dAzy_test-dAyz_test)
+(Axz-Azx)*(dAxz_test-dAzx_test)
+(Ayx-Axy)*(dAyx_test-dAxy_test)
)
-C*Tx*(dAzy_test-dAyz_test)
-C*Ty*(dAxz_test-dAzx_test)
-C*Tz*(dAyx_test-dAxy_test)
+(nud1x*(dAzy-dAyz)*(dAzy_test-
dAyz_test)
+nud1y*(dAxz-dAzx)*(dAxz_test-
dAzx_test)
+nud1z*(dAyx-dAxy)*(dAyx_test-
dAxy_test))
+I1x*(dAzy_test-dAyz_test)
+I1y*(dAxz_test-dAzx_test)
+I1z*(dAyx_test-dAxy_test),
```
and it can be formulated as
```
mu1*
(dFIx_test*FIx
+dFIy_test*FIy
+dFIz_test*FIz)+
(mud1x*dFIx_test*dFIx
+mud1y*dFIy_test*dFIy
+mud1z*dFIz_test*dFIz)
-mu1*C*
(dFIx_test*Tx
+dFIy_test*Ty
+dFIz_test*Tz)-
(dFIx_test*R1x
+dFIy_test*R1y
+dFIz_test*R1z)
```
when the reduced magnetic scalar potential is used.

Homogeneous Dirichlet boundary conditions have to be used for `dA`, and for `dFI` on the boundary segments, where Dirichlet boundary conditions are specified for `A` and for `FI`.

3.3. Matlab routine

After implementing the weak equations and boundary conditions, the arrangement must be discretized by a finite element mesh. After it, the problem must be saved as an `m` file for further use in Matlab.

Only some terms of the `fem.shape` variable must be modified by changing `shlag` by `shgp`, e.g. `shgp(3,4,''mu1'')`, where 3 means that the problem is three dimensional, and

4 is the Gaussian quadrature with four points. It should be as twice as the order of the approximation function. In this case `shlag(2,''FI'')` and `shcurl(2,{''Tx'',''Ty'',''Tz''})` have been used. This means that, nonlinear models are supposed in every Gaussian point of a finite element.

If `meshextend` has not run at the end of the `m` file, then the following command has to run:

`fem.xmesh=meshextend(fem);`

The Newton-Raphson algorithm implemented in Matlab is as follows:

```
while (error>EPS),
    step = step + 1;

    % 1.) dFI
    fem.sol = femstatic(fem, ...
        'u',fem.sol, ...
        'solcomp',{'dFI'}, ...
        'outcomp',{... }, ...
        'linsolver','gmres', ...
        'prefun','amg');

    % 2.) alpha
    alpha = 1;
    if lepes > 1,
        alpha = alfa_opt();
    end;

    % 3.) FI update -> fem.sol.u
    SOL         = fem.sol.u;
    SOL(rng_FI) = SOL(rng_FI) +
            alfa * SOL(rng_dFI);
    fem.sol     = femsol(SOL);

    % 4.) H,B,R,mud -> fem.sol.u
    fem = Nonlinear_step();

    % 5.) Test, error
    FUN_0 = Energy(fem);
    error = abs(FUN_0);
    ERROR(step) = error;
    ALFA(step)  = alfa;
end;
```

Here, `outcomp` contains all the variables have been inserted. The used solver is the `gmres` algorithm with `amg` preconditioner in these cases. The output of the solver is `dFI` or `dA`. The parameter `alpha` must be determined by the function `alfa_opt` if the value of `dFI` or `dA` is too large, then `FI` or `A` is updated in the variable `fem.sol.u`, which contains all the defined variables, and the range `rng_FI` selects only the appropriate values from this column vector, and `rng_FI = find(asseminit(fem, Init',{'FI',1}, Out','U'))`. The function `Nonlinear_step` contains the nonlinearity, which determines the magnetic flux density from the magnetic field intensity, or the magnetic field intensity from the magnetic flux density, as well as the differential permeability or the differential reluctivity in the case of Newton-Raphson method. The function `Energy` calculates the energy stored in the whole domain, which is minimized during the iteration. The nonlinear iteration is repeated until the `error` is small enough.

4. Numerical Model and Results

The nonlinearity, and its derivative, which is the permeability can be seen in Figure 2. The nonlinearity has been approached by piecewise linear approximation.
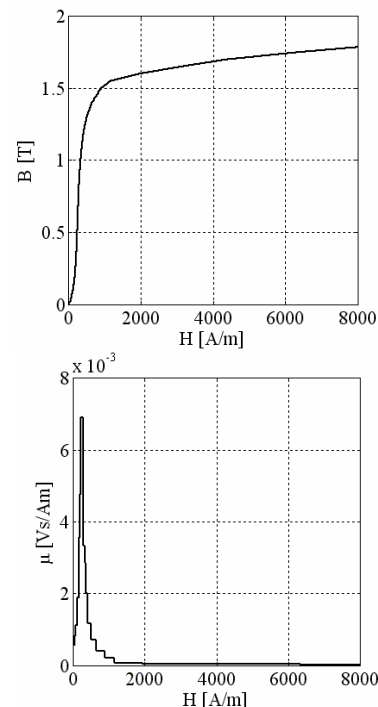

Figure 2. The nonlinear characteristics and the permeability.

4.1 TEAM 10

The problem can be seen in Figure 3. Steel plates have been placed around a racetrack
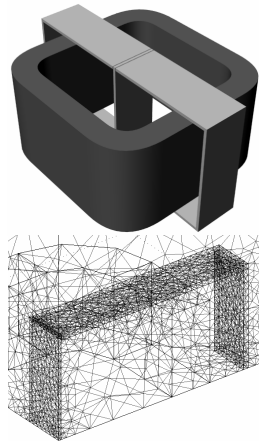
Figure 3. Problem TEAM 10 and the mesh.

shaped coil. Only the eigth of the problem can be analyzed, because of symmetry.

Two types of mesh have been tried out and analyzed. The first one consists of 8413 tetrahedra, the second one has been built up by 44828 finite elements. The first mesh results in 12850, 57278, and 43539 unknowns for Φ, vector A, and nodal A formulations, respectively. The second discretization results in 63807, 293814, and 218100 unknowns for the same formulations. The distribution of magnetic flux density vectors inside the plates can be seen in Figure 4. Figure 5 shows the magnetic flux density along a line placed inside the plates. The three formulations give practically the same results. It is noted that, the nodal A formulation is more sensitive to the density of the finite element mesh.
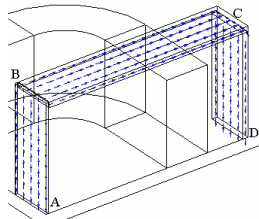
Figure 4. Magnetic flux is driven by the steels.
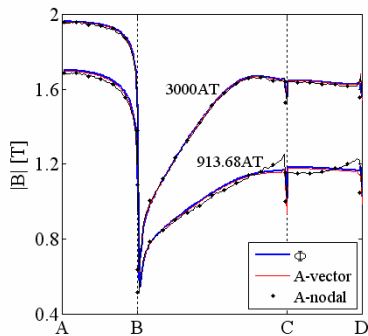
Figure 5. Magnetic flux by the potential formulations.

4.2 TEAM 13

This problem is a modified version of TEAM10. The U-shaped yokes have been translated as it can be seen in Figure 6. The magnetic flux density vectors are also plotted.
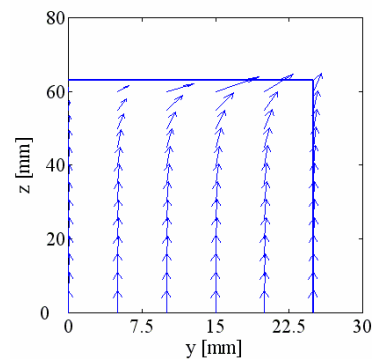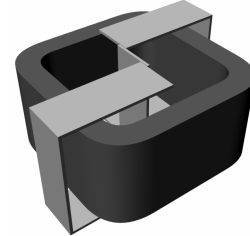

Figure 6. Problem TEAM 13, and the magnetic flux inside the central plate.

Figure 7 and Figure 8 show some comparisons between the results simulated by the three presented potential formulations. The magnetic flux density simulated by the reduced magnetic scalar potential is a little larger than the results obtained from the magnetic vector potential formulations, moreover the nodal vector potential formulation is more sensitive to the density of the mesh, as it was experienced in the last example, too. Comparison between measured and simulated data shows that the vector A formulation is the closest to the measured data.
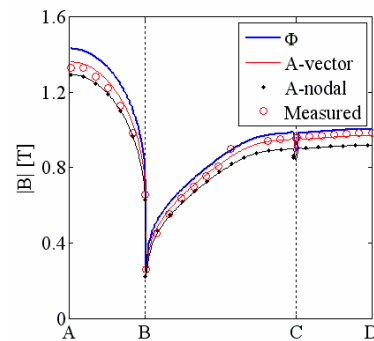

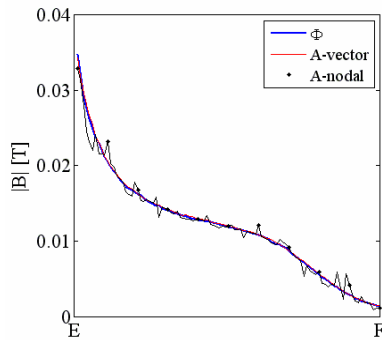**Figure 7.** Comparison of the magnetic flux density.

Figure 8. Comparison of the magnetic flux density under the horizontal plate.

4.3 TEAM 24

This is a modified version of the problem TEAM24 (Figure 9). The source current of the coils is constant, i.e. a static magnetic field problem has been analyzed, and the characteristics of the stator and the rotor is given by Figure 2.

The number of finite elements is 39060, which results in 55825 and 256530 unknowns for the Φ-formulation and for the edge element based *A*-formulation, respectively. The magnetic flux is mainly driven by the stator and the rotor steels as it can be seen in Figure 9, and a comparison between simulated results is shown in Figure 10.
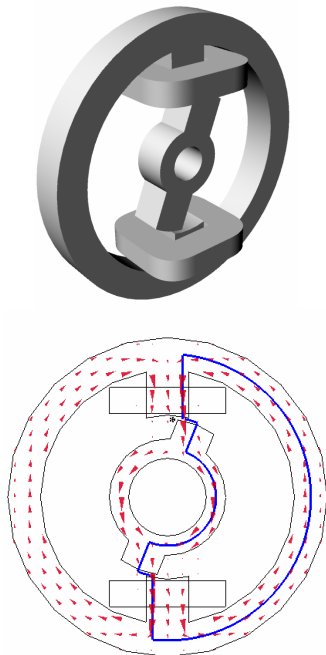

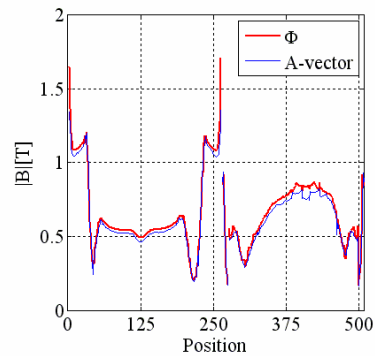**Figure 9.** Problem TEAM 24 and the magnetic flux density inside the rotor and stator.

Figure 10. Magnetic flux density along the path presented in Figure 9.

5. Conclusions

The proposed combination of the polarization technique and the Newton-Raphson method results in a much faster solver than the fixed point method, especially in the case of vector element representation of *A*. In the case of nodal formulation, under relaxation is almost always necessary, especially when the characteristics are very sharp, and the problem contains singularities.

The aim of further research is to implement the other potential formulations of the static magnetic field problems, and of the eddy current field problems, moreover an applicable hysteresis model based on Preisach's theory. The method must be tested on other problems as well.

6. References

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7. Acknowledgements